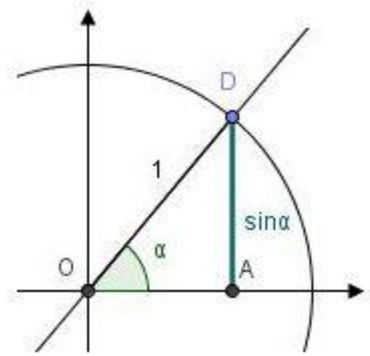
Standard Unit Circle and Significant Segments

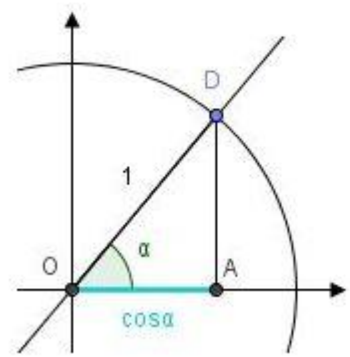
Defining trigonometric functions using significant segments

In the first quadrant

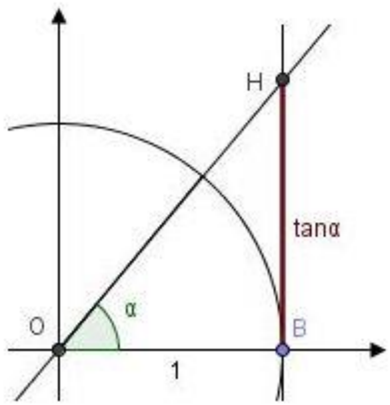
Sine of a angle



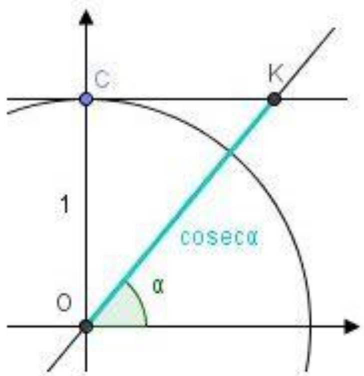
Cosine of an angle



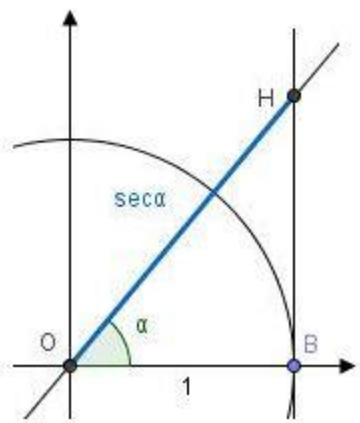
Tangent of an angle



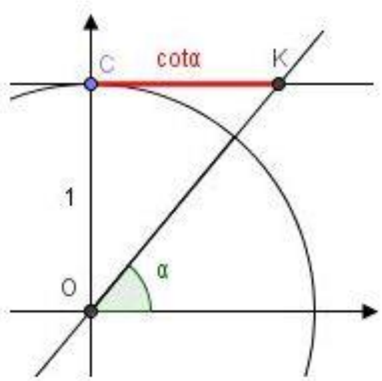
Cosecant of an angle



Secant of an angle

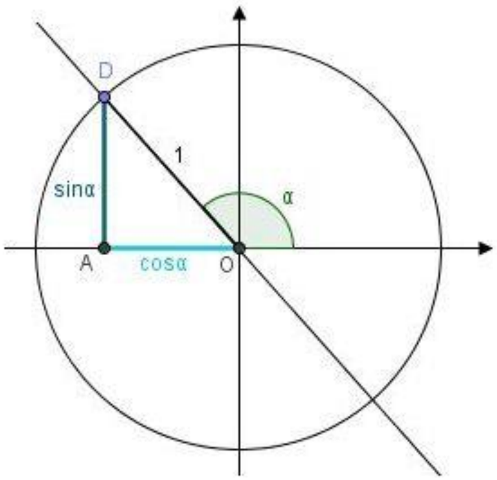


Cotangent of an angle

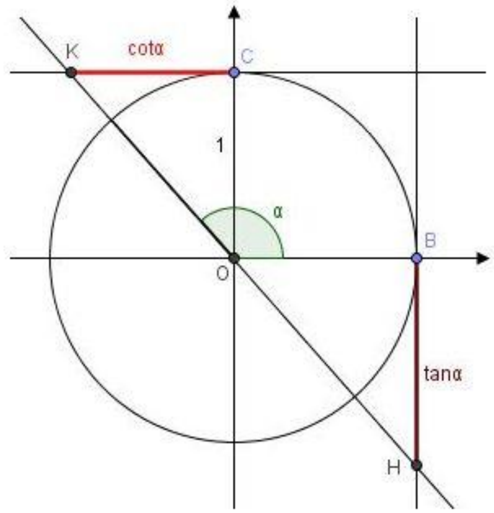


In the second quadrant

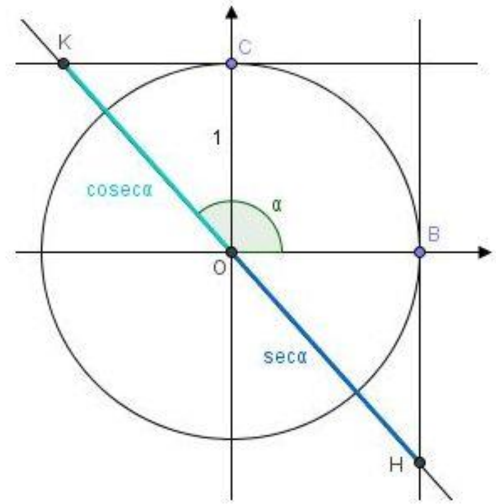
Sine and a cosine of an angle



Tangent and cotangent of an angle



Secant and cosecant of an angle



Relationships among trigonometric functions

Sine and cosecant of an angle

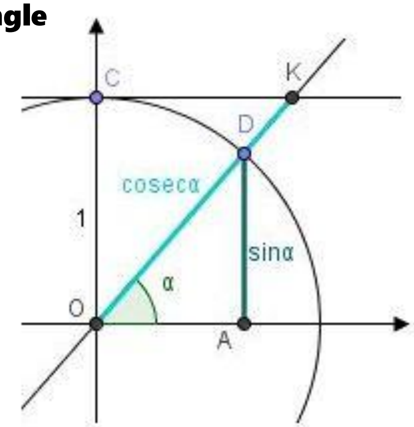
$\Delta OAD \sim \Delta KCO$

$$\frac{m\overline{AD}}{m\overline{OD}} = \frac{m\overline{CO}}{m\overline{OK}}$$

$$\frac{\sin\alpha}{1} = \frac{1}{\csc\alpha}$$

d'où :

$$\csc\alpha = \frac{1}{\sin\alpha}$$



Cosine and secant of an angle

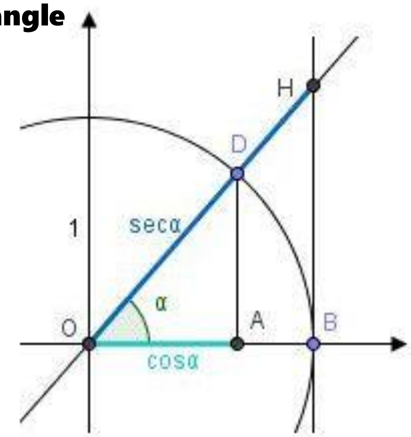
$\Delta OAD \sim \Delta OBH$

$$\frac{m\overline{OA}}{m\overline{OD}} = \frac{m\overline{OB}}{m\overline{OH}}$$

$$\frac{\cos\alpha}{1} = \frac{1}{\sec\alpha}$$

d'où :

$$\sec\alpha = \frac{1}{\cos\alpha}$$



Sine, cosine and tangent of an angle

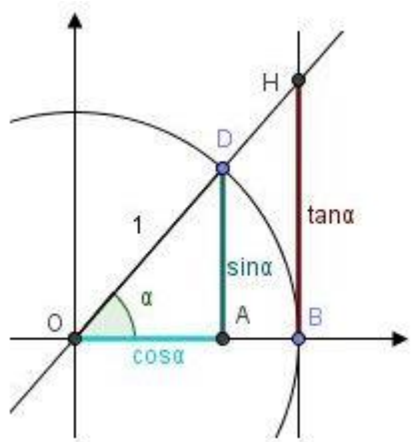
$\Delta OAD \sim \Delta OBH$

$$\frac{m\overline{AD}}{m\overline{OA}} = \frac{m\overline{BH}}{m\overline{OB}}$$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\tan\alpha}{1}$$

d'où :

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$



Sine, cosine and cotangent of an angle

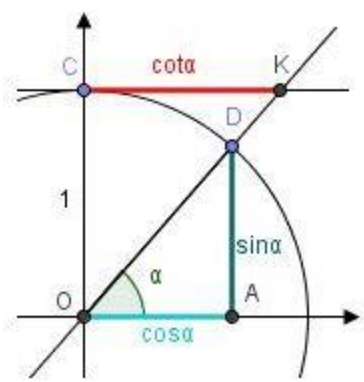
$\Delta OAD \sim \Delta KCO$

$$\frac{m\overline{OA}}{m\overline{AD}} = \frac{m\overline{KC}}{m\overline{CO}}$$

$$\frac{\cos\alpha}{\sin\alpha} = \frac{\cot\alpha}{1}$$

d'où :

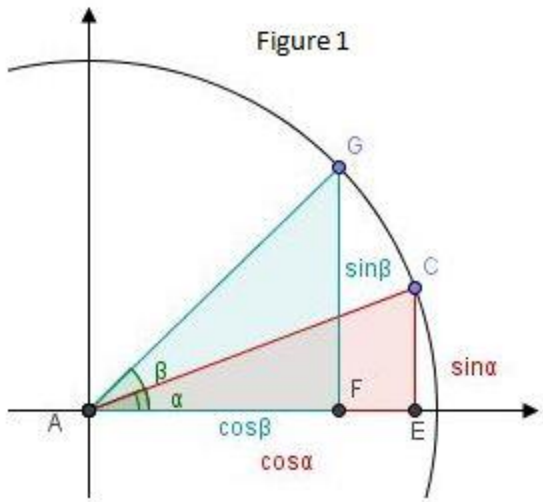
$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$$



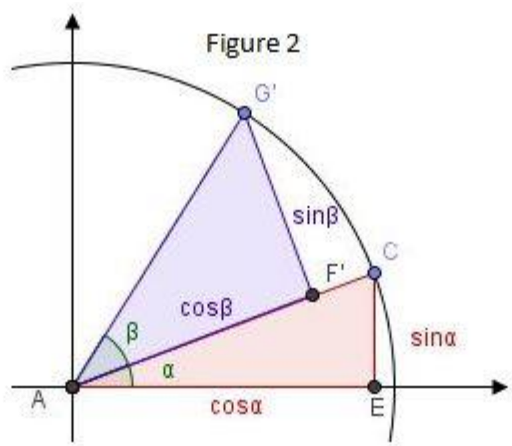
Sums and difference of two angles

Sine of the sum of two angles

In Figure 1, angle α determines Δ AEC and angle β determine Δ AFG.



In Figure 2, Δ AF'G' is the image of Δ AFG after it was rotated about center A and angle α.



In order to calculate sin ($\alpha+\beta$), we must determinate the lengtj of segment JG′ of Δ AJG′ in Figure 3.

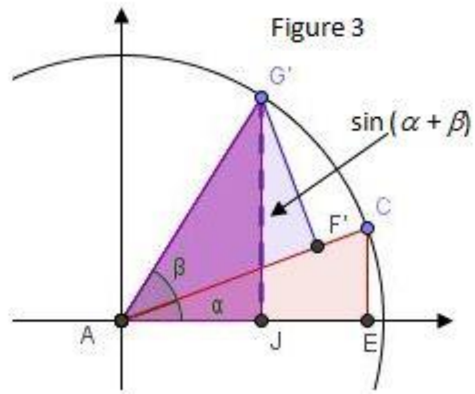
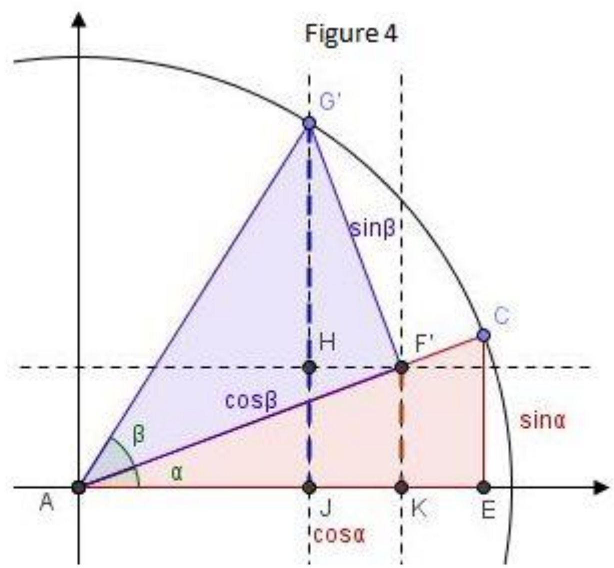


Figure 4 shows that this segment can be broken down into the sum of segments JH and HG′ and that segments JH et KF′ are congruent. In addition, we can show that Δ G′HF′ and Δ AKF′ are similar.

From the information above, we can deduce that :



$$\sin\alpha = \frac{m\overline{KF'}}{\cos\beta} = \frac{m\overline{JH}}{\cos\beta}$$

$$\sin\alpha\cdot\cos\beta = m\overline{JH}$$

and that :

$$\cos\alpha = \frac{m\overline{G'H}}{\sin\beta}$$

$$\sin\beta\cdot\cos\alpha = m\overline{G'H}$$

therefore :

$$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta$$

Cosine of the sum of two angles

In order to calculate cos ($\alpha + \beta$), we must determine the length of segment AJ of ΔAJG' in Figure 5.

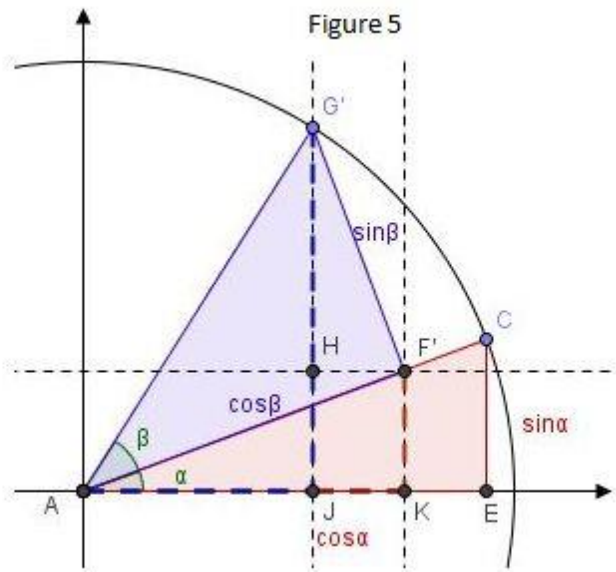
The length of this segment is the result of the difference between the lengths of segments AK et JK. In addition, as shown in Figure 5, segments JK and HF' are congruent.

In addition, as in Figure 4, we can show that ΔG'HF' and ΔAKF' are similar.

From the information above, we can deduce that :

$$\cos\alpha = \frac{m\overline{AK}}{\cos\beta}$$

$$\cos\alpha \cdot \cos\beta = m\overline{AK}$$



and that :

$$\sin\alpha = \frac{m\overline{HF'}}{\sin\beta} = \frac{m\overline{JK}}{\sin\beta}$$

$$\sin\alpha \cdot \sin\beta = m\overline{JK}$$

therefore :

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$

Difference of two angles

We can show that sin ($-\alpha$) = – sin α and that cos ($-\alpha$) = cos α

therefore :

$$\sin(\alpha - \beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$$

and

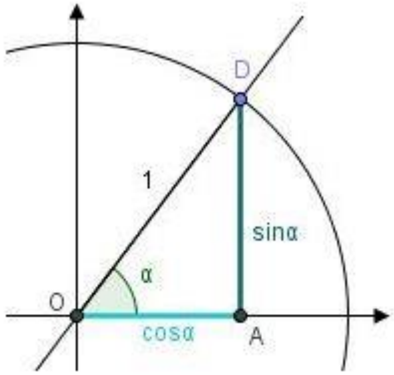
$$\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$$

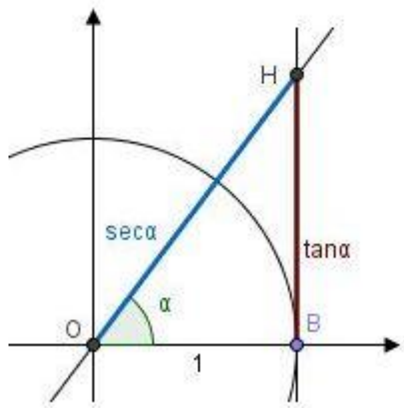
Pythagorean identities

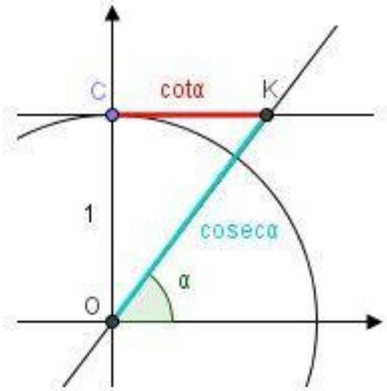
$\sin^2\alpha + \cos^2\alpha = 1$

$1 + \tan^2\alpha = \sec^2\alpha$

$1 + \cot^2\alpha = \text{cosec}^2\alpha$

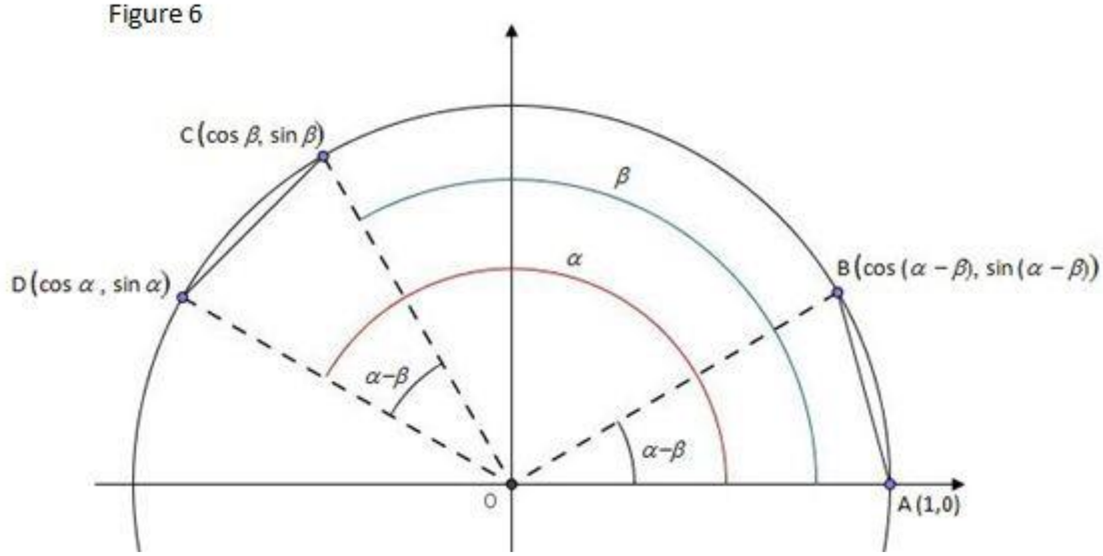






Cosine of the difference of two angles

Figure 6 shows the difference between angles α and β. In addition, $m\overline{AB} = m\overline{CD}$.



By usinng the relationship governing the distance between two points, i.e. $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, nwe can deduce that:

$$\left(m\overline{AB}\right)^2 = \left(\cos(\alpha-\beta)-1\right)^2 + \left(\sin(\alpha-\beta)-0\right)^2$$
$$\left(m\overline{AB}\right)^2 = \cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)$$
$$\left(m\overline{AB}\right)^2 = \left(\cos^2(\alpha-\beta) + \sin^2(\alpha-\beta)\right) - 2\cos(\alpha-\beta) + 1$$
$$\left(m\overline{AB}\right)^2 = 1 - 2\cos(\alpha-\beta) + 1$$
$$\left(m\overline{AB}\right)^2 = 2 - 2\cos(\alpha-\beta)$$

and that :

$$\left(m\overline{CD}\right)^2 = \left(\cos\beta - \cos\alpha\right)^2 + \left(\sin\beta - \sin\alpha\right)^2$$
$$\left(m\overline{CD}\right)^2 = \cos^2\beta - 2\cos\alpha\cos\beta + \cos^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta + \sin^2\alpha$$
$$\left(m\overline{CD}\right)^2 = \left(\cos^2\beta + \sin^2\beta\right) + \left(\cos^2\alpha + \sin^2\alpha\right) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$$
$$\left(m\overline{CD}\right)^2 = 1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$$
$$\left(m\overline{CD}\right)^2 = 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$$

therefore, since $\left(m\overline{AB}\right)^2 = \left(m\overline{CD}\right)^2$,

$$2 - 2\cos(\alpha-\beta) = 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$$
$$-2\cos(\alpha-\beta) = -2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$