The Study of Matrices in the *Technical and Scientific* Option

In the Secondary Cycle Two mathematics component of the Québec Education Program, the program content for the *Technical and Scientific* option refers more than once to the possible use of matrices. For example, page 82 states that: "*In the last year of the cycle, students also learn to use matrices, which gives them the opportunity to expand and consolidate their knowledge of mathematics. In dealing with meaningful situations involving operations on matrices, students will become aware of how effective this register is in processing data.*"

Matrices are specifically introduced as a register of representation used to interpret, process and manipulate several data items at a time. Basic operations such as addition (subtraction) and multiplication (by a scalar and of matrices) could be presented (e.g.: purchases/sales, inventory). Systems of equations could be solved using an augmented matrix and the reduction method. The use of the previously studied geometric transformations (reflection, translation, rotation, dilatation) and their representation in the form of matrices make it possible to apply and consolidate knowledge of these transformations by using concepts and processes associated with analytic geometry and trigonometry. (A rotation could be performed using the measures of significant angles.) In addition, connections could me made with certain technological applications (e.g.: spreadsheets, computer graphics).

Matrices are introduced and integrated into the study of the different branches of mathematics. Students begin learning about them through situations where the use of matrices is appropriate and the related vocabulary is introduced when necessary. This document reviews the main elements associated with matrices and matrix operations as well as a few examples of their application.

MATRICES

An $r \times c$ matrix is a rectangular table of rc elements arranged in r rows and c columns.

Example: here is a 2 x 3 matrix: $\begin{pmatrix} 4 & 7 & -9 \\ -3 & \frac{1}{2} & 1.5 \end{pmatrix}$

Row matrix: a matrix with only one row — Example: $\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$

Column matrix: a matrix with only one column — Example: $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

Square matrix: a matrix with the same number of columns and rows

Examples: $\begin{pmatrix} 2 & 7 \\ 9 & -1.5 \end{pmatrix}$ or $\begin{pmatrix} 4 & -3 & 8 \\ 6 & 1 & 0 \\ -2 & 0 & 5 \end{pmatrix}$

OPERATIONS ON MATRICES

Addition and subtraction

To add or subtract two matrices:

1. Check if they have the same dimensions.
2. Add or subtract the elements in the corresponding positions and indicate the result in the corresponding position in the resulting matrix.

Example:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{pmatrix} = \begin{pmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ d_1+d_2 & e_1+e_2 & f_1+f_2 \end{pmatrix}$$

Multiplication

To multiply two matrices:

1. Check if the number of columns in the first matrix is equal to the number of rows in the second matrix. The product of matrix $m \times n$ and matrix $n \times p$ will be matrix $m \times p$.
2. To multiply a row matrix by a column matrix, multiply the corresponding elements, then add the products.

Examples:

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} ad + be + cf \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Properties of multiplication

Multiplication is associative, but not commutative.

Matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element for the multiplication of 2×2 matrices:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

USING MATRICES TO REPRESENT AND INTERPRET A SITUATION

The ability to mathematize is also developed in using matrices to record data, make data tables or record information. For example, a salesperson could record his weekly sales as follows:

$$\begin{pmatrix} 9 & 3 & 0 & 5 & 3 & 8 \\ 6 & 0 & 0 & 3 & 0 & 6 \\ 6 & 6 & 0 & 1 & 3 & 6 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}.$$

Each column represents a day of the week (M, T, W, T, F, S) and each row represents the sales made in a given week.

However, producing a matrix does not necessarily lead to the development of mathematical abilities unless it involves processing the information mathematically and interpreting the results. For example, a shoe salesperson's inventory at the end of September could be presented as follows:

September		Types		
		A	B	C
Sizes	34	4	2	3
	36	1	5	2
	38	3	1	5

In October, he purchased:

		Types		
		A	B	C
Sizes	34	1	3	1
	36	4	0	1
	38	0	2	0

During the month, he sold:

		Types		
		A	B	C
Sizes	34	3	2	2
	36	2	1	1
	38	1	0	1

What is his inventory at the end of October? What is the value of his merchandise given that he pays $45 for a type A shoe, $38 for a type B shoe and $25 for a type C shoe and that he sells a type A shoe for $60, a type B shoe for $55 and a type C shoe for $40. How much profit does he make?

Inventory

$$\begin{pmatrix} 4 & 2 & 3 \\ 1 & 5 & 2 \\ 3 & 1 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 4 \\ 5 & 4 & 3 \\ 3 & 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 5 & 4 \\ 5 & 4 & 3 \\ 3 & 3 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 3 & 2 \\ 2 & 3 & 4 \end{pmatrix}$$

Monthly sales

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 60 \\ 55 \\ 40 \end{pmatrix} = \begin{pmatrix} 3\times 60+2\times 55+2\times 40 \\ 2\times 60+1\times 55+1\times 40 \\ 1\times 60+0\times 55+1\times 40 \end{pmatrix} = \begin{pmatrix} 370 \\ 215 \\ 100 \end{pmatrix}$$

for a total of $685

Value of the merchandise

$$\begin{pmatrix} 2 & 3 & 2 \\ 3 & 3 & 2 \\ 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 60 \\ 55 \\ 40 \end{pmatrix} = \begin{pmatrix} 2\times 60+3\times 55+2\times 40 \\ 3\times 60+3\times 55+2\times 40 \\ 2\times 60+3\times 55+4\times 40 \end{pmatrix} = \begin{pmatrix} 365 \\ 425 \\ 445 \end{pmatrix}$$

Total value: $1235

Monthly purchases

$$\begin{pmatrix} 1 & 3 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 45 \\ 38 \\ 25 \end{pmatrix} = \begin{pmatrix} 1\times 45+3\times 38+1\times 25 \\ 4\times 45+0\times 38+1\times 25 \\ 0\times 45+2\times 38+0\times 25 \end{pmatrix} = \begin{pmatrix} 184 \\ 205 \\ 76 \end{pmatrix}$$

For a total of $465

Profit : $685 – $465 = $220

SYSTEMS OF EQUATIONS

Starting in the first year of Secondary Cycle Two, students use different methods to determine the solution of a system of equations. The introduction of matrices provides students with the opportunity to use another problem-solving method, namely reducing systems of equations to the simplest possible equivalent.

The linear transformations (also called linear combinations) that can be used to develop equivalent systems of equations are as follows:

- Multiplying (dividing) an equation by a scalar;
  in multiplying (1) by 2, we obtain a system that is equivalent to the first:

$$\begin{array}{l} x+4y=24 \ (1) \\ 2x+y=16 \ (2) \end{array} \quad \text{is equivalent to} \quad \begin{array}{l} 2x+8y=48 \ (3) \\ 2x+y=16 \ (4) \end{array}$$

- Adding (subtracting) two equations;
  in subtracting (4) from (3), we obtain a new system that is equivalent to the first:

$$\begin{array}{r} 7y=32 \ (5) \\ 2x+y=16 \ (6) \end{array}$$

A quick glance at the basic transformations involved shows that it would be better to use only the coefficients and a matrix representation. Here are the augmented matrices associated with the three preceding systems:

$$\left(\begin{array}{cc|c} 1 & 4 & 24 \\ 2 & 1 & 16 \end{array}\right) \quad \left(\begin{array}{cc|c} 2 & 8 & 48 \\ 2 & 1 & 16 \end{array}\right) \quad \left(\begin{array}{cc|c} 0 & 7 & 32 \\ 2 & 1 & 16 \end{array}\right).$$

Applying the basic transformations to find the simplest equivalent system of equations results in the following intermediate systems of equations and the corresponding augmented matrices:

- in multiplying (6) by 7

$$\begin{array}{r} 7y = 32 \ (7) \\ 14x+7y=112 \ (8) \end{array} \quad \left(\begin{array}{cc|c} 0 & 7 & 32 \\ 14 & 7 & 112 \end{array}\right)$$

- in subtracting (7) from (8)

$$\begin{array}{r} 7y = 32 \ (9) \\ 14x=80 \ (10) \end{array} \quad \left(\begin{array}{cc|c} 0 & 7 & 32 \\ 14 & 0 & 80 \end{array}\right)$$

- in dividing (9) by 7 and (10) by 14

$$\begin{array}{l} y=\frac{32}{7} \ (11) \\ x=\frac{40}{7} \ (12) \end{array} \quad \left(\begin{array}{cc|c} 0 & 1 & \frac{32}{7} \\ 1 & 0 & \frac{40}{7} \end{array}\right) \text{ ou } \left(\begin{array}{cc|c} 1 & 0 & \frac{40}{7} \\ 0 & 1 & \frac{32}{7} \end{array}\right)$$

Students can therefore solve systems of equations in different ways: using augmented matrices, tables of values, graphs, algebra (comparison, substitution and reduction methods). They choose the method that is best suited to the system in question.

TRANSFORMATIONAL GEOMETRY

Matrices can be used in geometry since the algebraic notation of matrices makes it possible to study geometric transformations. A knowledge of reflections, translations, rotations, dilatations, analytic geometry and trigonometry makes it possible to explore matrices in greater detail and to construct a matrix representation of isometries and similarity transformations.

Point (x, y) in the Cartesian plane can be represented by the following two column matrices:

$$\begin{pmatrix} x \\ y \end{pmatrix} \text{ or } \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

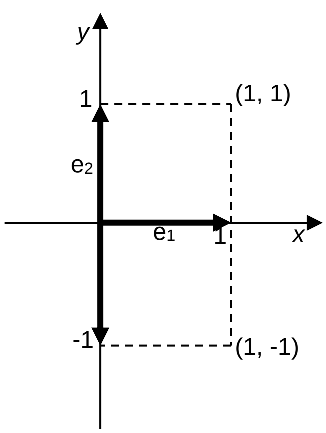
Matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, associated with vector combination $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, is the identity element for the multiplication of 2 × 2 matrices:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

To identify a transformation that can be represented by a 2 × 2 matrix, simply determine the images of points (1, 0) and (0, 1) or of vectors e_1 and e_2.

Note: For all transformations represented by 2 × 2 matrices, the origin is its own image.

A reflection about the x-axis matches the ordered pair (x, y) with the ordered pair (x, $-y$).



In this case, vector $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is transformed into $S_x(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

and vector $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is transformed into $S_x(e_2) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

The transformation matrix is $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

A simple calculation shows that the product $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$ matches (x, y) with (x, $-y$).

As well, matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is associated with the reflection about the *y*-axis and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ the central symmetry (or a rotation of 180° about the origin or a dilatation with a scale factor of –1, centred at the origin).

Example:

Triangle ABC, where A = (4, 2), B = (–3, 2) and C = (1, –7), can be represented by the following matrix:

$$\begin{pmatrix} 4 & -3 & 1 \\ 2 & 2 & -7 \end{pmatrix}$$

To determine the image of triangle ABC under a reflection about the *x*-axis, calculate the following product:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4 & -3 & 1 \\ 2 & 2 & -7 \end{pmatrix} = \begin{pmatrix} 4 & -3 & 1 \\ -2 & -2 & 7 \end{pmatrix}$$

To determine the matrix of the composite of two transformations, simply multiply the matrices associated with these transformations.

Example:

The matrix representing the composite of a reflection about the *y*-axis followed by a rotation of 90˚ about the origin is as follows:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For any 2 × 2 matrix, there is a corresponding 3 × 3 matrix:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To use 3 × 3 matrices, represent point (*x*, *y*) by the following matrix:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Since the origin is not fixed under a translation, a translation cannot be represented by a 2 × 2 matrix, but this can be done using a 3 × 3 matrix.

Translation $t_{(h,k)} : (x, y) \mapsto (x+h, y+k)$ can be represented by the following matrix:

$$\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

It is also easy to discover that the matrix for a dilatation centred at the origin and with a scale factor of *k* can be represented by the following matrix:

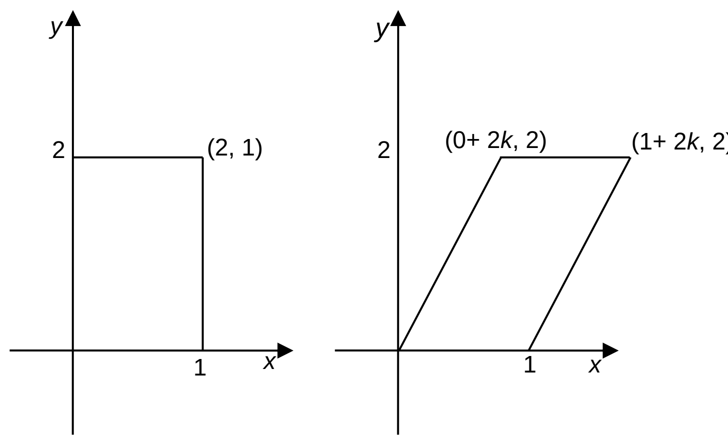
$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \text{ou} \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and by $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in the case of an expansion or a scale change.

The matrix for a rotation of angle θ centred at the origin is:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

"Geometric transformations are also used in a number of activities. For instance, they can be used to . . . construct the image of a figure from a transformation matrix."[1] In addition to the geometric transformations covered in the program, students could also explore other transformations, such as shearing used in graphic animation.



A geometric figure that undergoes horizontal shearing will be deformed, as though the *y*-axis were slanted towards the right or left depending on whether *k* is positive or negative. However, the horizontal lines remain horizontal and vertical in the case of vertical shearing.

Transformation $C_{x;k} : (x, y) \mapsto (x+ky, y)$ can be represented by the matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

1. *Québec Education Program*, *Secondary Cycle Two*, 93

CONNECTION WITH TECHNOLOGY

Matrices are an integral part of many technological tools.

Examples:

In a spreadsheet, calculating the number of points scored by hockey teams (column F) involves using a formula that is the same for each team.

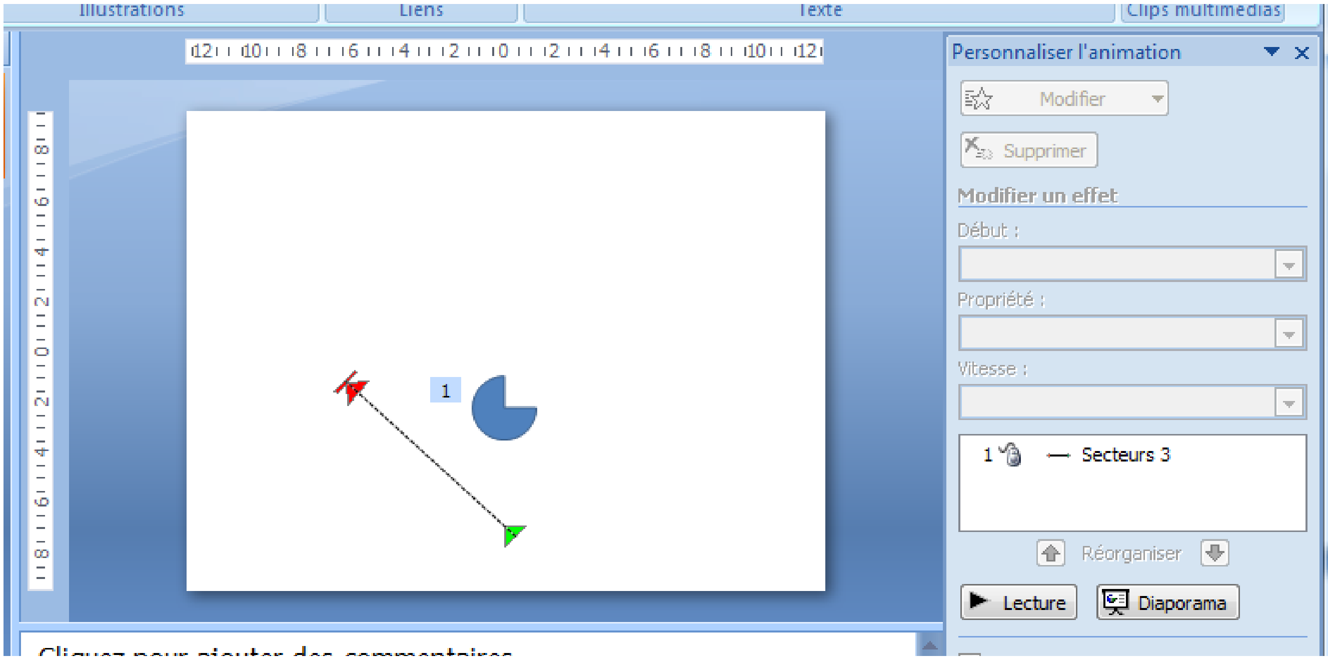
F2 | =C2*2+D2*0+E2*1

	A	B	C	D	E	F	G	H
1	**Équipes**	**PJ**	**V**	**D**	**DP**	**Pts**		
2	Wa	27	16	5	6	38		
3	Pi	28	19	9	0	38		
4	Bu	24	15	7	2	32		
5	NJ	24	17	6	1	35		
6	At	24	14	7	3	31		
7	Bo	26	13	8	5	31		
8	Ot	25	13	8	4	30		
9	TB	25	10	7	8	28		
10	Ph	24	13	10	1	27		
11	NYR	27	13	13	1	27		
12	NYI	27	10	10	7	27		
13	Mo	27	12	13	2	26		
14	Fl	26	10	12	4	24		
15	To	26	7	12	7	21		
16	Ca	27	5	17	5	15		
17								

This calculation can be represented by the product of two matrices:

$$\begin{pmatrix} 16 & 5 & 6 \\ 19 & 9 & 0 \\ 15 & 7 & 2 \\ 17 & 6 & 1 \\ 14 & 7 & 3 \\ 13 & 8 & 5 \\ 13 & 8 & 4 \\ 10 & 7 & 8 \\ 13 & 10 & 1 \\ 13 & 13 & 1 \\ 10 & 10 & 7 \\ 12 & 13 & 2 \\ 10 & 12 & 4 \\ 7 & 12 & 7 \\ 5 & 17 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 38 \\ 38 \\ 32 \\ 35 \\ 31 \\ 31 \\ 30 \\ 28 \\ 27 \\ 27 \\ 27 \\ 26 \\ 24 \\ 21 \\ 15 \end{pmatrix}$$

In presentation software, the position of an object is determined by the coordinates of the points (pixels) that compose it. An animation, for instance, a leftward path, consists in moving an object under a translation. This displacement, which consists of a series of positions at different points in time, is carried out using matrix operations.



Instantaneous displacement is the result of the product of the matrix representing the position of the object and the translation matrix.

REFERENCES OR TO FIND OUT MORE

Québec. Ministère de l'Éducation, du Loisir et du Sport. *Québec Education Program*, Secondary Cycle Two, Chapter 6, Mathematics, Science and Technology. Québec: Gouvernement du Québec, 2007.
[Online]
http://www.mels.gouv.qc.ca/sections/programmeFormation/secondaire2/medias/en/Mathematique_SecondVersion_a.pdf

Québec. Ministère de l'Éducation. *Guide pédagogique : Mathématique, secondaire, second cycle, Fascicule A - L'arithmétique et l'algèbre*. Québec: Gouvernement du Québec, 1984.

Manitoba, Department of Education. *Programme d'études : document de mise en œuvre, Mathématiques appliquées, secondaire 4, Unité A - Modèles matriciels.* Manitoba: Department of Education, 2001.
[Online] http://www.edu.gov.mb.ca/m12/frpub/ped/ma/app40sdmo/index.html

http://www.philipperey.net/maths3-4/algebreLineaireVecteurs/algebreLineaire/aQuoiCaSert.php

http://www.cs.laurentian.ca/jdompierre/html/MATH1057F_W2009/cours/s1.9_MatriceTransformation.pdf

http://www.cegep-ste-foy.qc.ca/profs/lgingras/documents/infographie_et_matrices.pdf