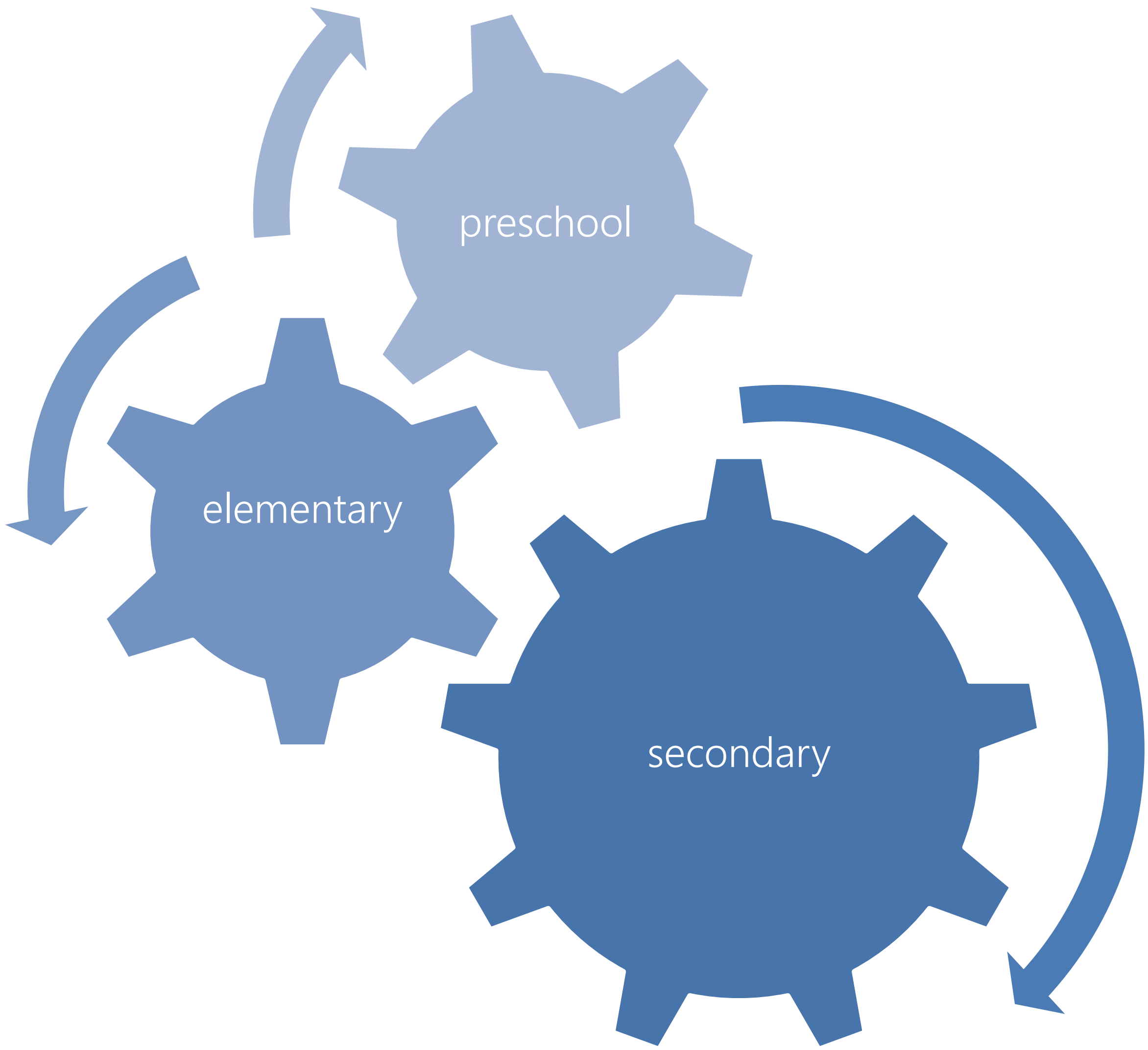
Harmonization of Learning Cycles

Mathematics

Companion Document



Spring 2011

HARMONIZATION OF LEARNING CYCLES
COMPANION DOCUMENT

This document contains passages (in bold type) from the Québec Education Program and the Progression of Learning – Mathematics as well as suggestions, clarifications, examples and illustrations related to:

- the development of concepts and processes
- the construction of networks of concepts and processes
- the meaning of operations involving numbers:
  - additive structures
  - multiplicative structures
- operations involving numbers:
  - personal and conventional processes
  - mental computation
  - use of technology
- manipulation

Mathematics is a science that involves abstract concepts and language. Students develop their mathematical thinking gradually through personal experiences and exchanges with peers. Their learning is based on situations that are often drawn from everyday life. In elementary school, students take part in learning situations that allow them to use objects, manipulatives, references and various tools and instruments. The activities and tasks suggested encourage them to reflect, manipulate, explore, construct, simulate, discuss, structure and practise, thereby allowing them to assimilate concepts, processes and strategies that are useful in mathematics. Students must also call on their intuition, sense of observation, manual skills as well as their ability to express themselves, reflect and analyze. By making connections, visualizing mathematical objects in different ways and organizing these objects in their minds, students gradually develop their understanding of abstract mathematical concepts. (*Progression of Learning in Secondary School – Mathematics*, p. 5)

Teachers should adopt a pedagogical approach that takes into account the emergence of reflective thought processes in students, providing the mental stimulus for them to think about the concepts that they are learning. This approach should also take student interactions into account, going as far as creating a community of learners because students consolidate their learning when they work on the same concepts with their peers. The teacher also needs to put well- chosen learning models and tools in place to support the development of mathematical concepts in students.

In this way, students build a set of tools that will allow them to communicate[1] appropriately using mathematical language, reason effectively by making connections between mathematical concepts and processes, and solve situational problems. Emphasis is placed on technological tools, as these not only foster the emergence and understanding of mathematical concepts and processes, but also enable students to deal more effectively with various situations. By using mathematical concepts and various strategies, students can make informed decisions in all areas of life. Combined with learning activities, the situations experienced by students promote the development of mathematical skills and attitudes that allow them to mobilize, consolidate and broaden their mathematical knowledge. (*Progression of Learning in Elementary School – Mathematics*, p. 3)

[1] Taking the cross-curricular competency *To communicate appropriately* one step further, the competency *To communicate by using mathematical language* requires that students learn and coordinate the elements of mathematical language (types of representation) that are used in conceptualizing mathematical objects. When using these mathematical concepts and processes, students must interpret or produce mathematical messages.

THE DEVELOPMENT OF CONCEPTS AND PROCESSES

Reasoning in mathematics consists in establishing relationships, combining them and using them to perform a variety of operations in order to create new concepts and take one's mathematical thinking to a higher level. (*Québec Education Program*, p. 140) To reason is to logically organize a series of facts, ideas or concepts in order to arrive at a conclusion that should be more reliable than one resulting from an impression or intuition. Mathematical reasoning involves apprehending the situation, mobilizing relevant concepts and processes and making connections. In so doing, the students become familiar with mathematical language, construct the meaning of mathematical concepts and processes and establish links between them. (*Québec Education Program*, p. 144)

What is a concept?

A concept is an idea or mental image that corresponds to some distinct class of entities, whether mathematical or not.

Once students demonstrate their ability to consistently give examples and counterexamples and identify the presence or absence of an element of the concept, they have acquired the ability to conceptualize. Here are a few of the ways in which students demonstrate that they have acquired and mastered a concept:

- by naming the concept, talking about it and defining it
- by recognizing and producing examples and counterexamples of the concept
- by using various types of representation (drawing, diagram, graph, symbol) to represent the concept and making transfers among these representations
- by identifying the essential attributes of the concept
- by comparing concepts that have a certain affinity, identifying their common as well as their differentiating attributes

Building a concept and understanding it

Let us make the following analogy: building a concept is like building an object. To build an object, we need tools, materials and effort. Building a concept follows a similar pattern. John A. Van de Walle and LouAnn H. Lovin argue that the tools we use to build our understanding are our preliminary or prior ideas, in other words, the knowledge we already possess. Our materials are what we see, hear or touch, in other words, the elements of our physical environment. Sometimes these materials will be our ideas or our thoughts, in other words, the ideas we already have and the thoughts that lead us to modify some of them. Finally, the effort required is an active and reflective thought process. There can be no effective learning unless our spirit is engaged in a process of reflection.[2]

Conceptualization occurs when links are forged between the concept to be acquired and earlier ideas related to that concept. Because these earlier ideas impart meaning to the new concept, new links are formed between the new and previous concepts. The more the earlier concepts are used to give meaning to the new concept, the more links are created and the more links that are created, the better the understanding of the new concept will be. The level of understanding will be the measure of the quality and quantity of links connecting the new concept to prior concepts.

Van de Walle and Lovin claim that understanding varies in accordance with the existence of relevant ideas and the creation of new links.[3]

[2] John A. Van de Walle and LouAnn H. Lovin, *Teaching Student-Centered Mathematics* (Allyn & Bacon, 2006), Volume 1: Grades K-3.

[3] Ibid.

What is a process?

A process is an ordered series of rules to be applied against data to arrive with certainty at a definite result, regardless of the data. For example, a process of addition is a series of actions, a step, which enables us to obtain the result of adding two numbers.

The following table provides an overview of the criteria showing that students have learned and mastered a concept, a process or a strategy.

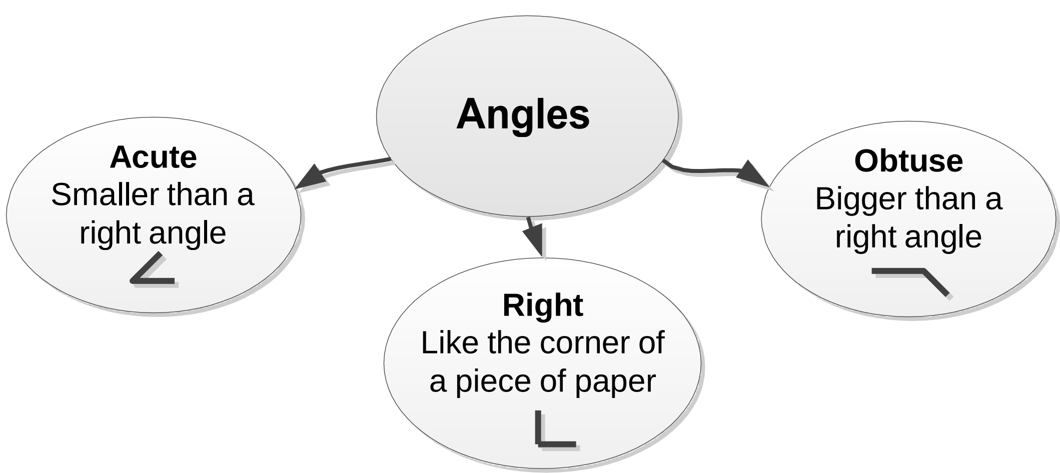
I have learned a concept if . . .	I have learned a process or a strategy if . . .
• I can identify its essential qualities (properties). • I can produce examples and counterexamples. • I can communicate a personal definition. • I can relate the concept to other concepts. • I can recognize the concept in a situation.	• I can produce a description, definition or example of the process. • I can understand its importance and purpose. • I know how to implement it. • I know and can explain all the steps involved in implementing it. • I can compare my process or strategy with other processes or strategies. • I can use concepts and properties to explain the steps in my process or strategy. • I know when to use it.

NETWORKS OF CONCEPTS AND PROCESSES

During Cycle One, the students work at putting together a network of mathematical concepts and processes. (*Québec Education Program*, p. 145)

A network of concepts and processes represents the links connecting the concepts and processes themselves. Networks of concepts and processes that have a wealth of links and are well constructed by students, demonstrate how thoroughly they understand the illustrated concepts. The networks are dynamic and personal to each student. They become enriched as learning goes on.

Here is an example of a network of concepts involving angles in Elementary Cycle Two:



MEANING OF OPERATIONS

In order to fully understand operations and their different meanings in various contexts, students must understand the relationships among data and among operations, and choose and perform the correct operations, taking into account the properties and order of operations. Students must also have a general idea of the result expected.

Students will thus be encouraged to use concrete, semi-concrete or symbolic means to mathematize a variety of situations illustrating different meanings. In these situations, students will learn to break problems down into simpler ones and identify the relationships among data that will help them to arrive at a solution. Since operation sense is developed at the same time as number sense, the two should be taught concurrently. (*Progression of Learning in Elementary School*, p. 9)

In addition or subtraction, quantities are added, taken away, united or compared. It is important that students use the various meanings of addition and subtraction to work out all kinds of problems.

The problems presented in the following table appear to be similar. For students, however, each situation represents a specific problem. It is in mastering these different types of problems that students acquire a mastery of addition and subtraction.

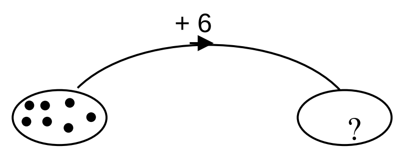
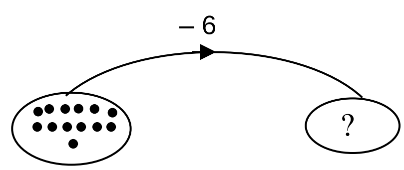
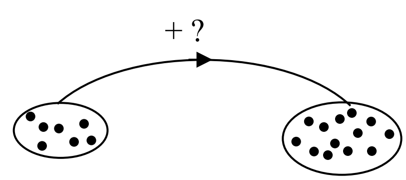
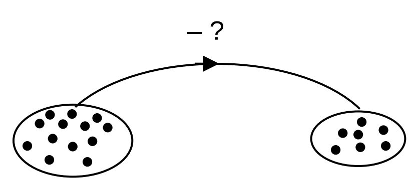
Multiplication is the gathering together of objects that were in equal-sized groups while division is the dividing of objects into equal-sized groups. To develop the meaning of these two operations, the three quantities underpinning multiplication and division have to be recognized: the total amount, the number of equal groups and the size of each group. Once again, it is important to have students work on a wide range of problems that present the various meanings of multiplication and division.

The Additive structures and Multiplicative structures tables on the following pages present the different meanings of addition and subtraction as well as those of multiplication and division that, according to the Progression of Learning, are to be explored with the students.

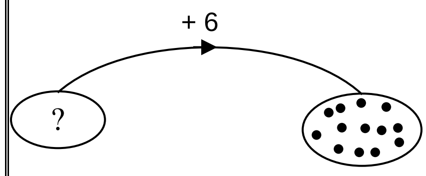
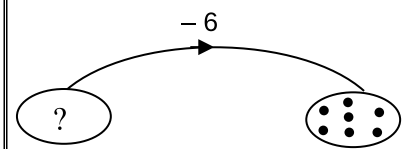
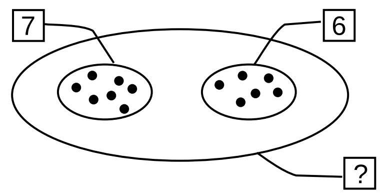
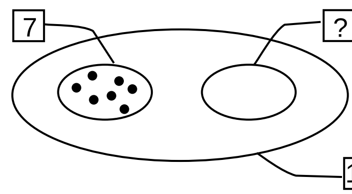
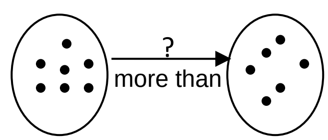
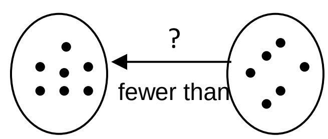
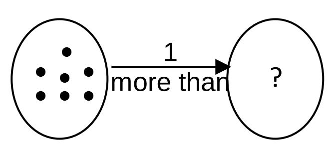
ADDITIVE STRUCTURES

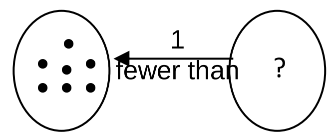
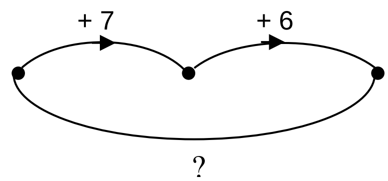
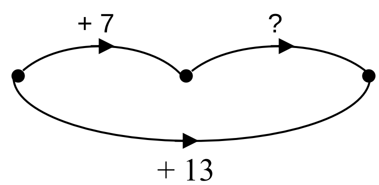
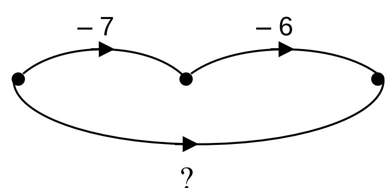
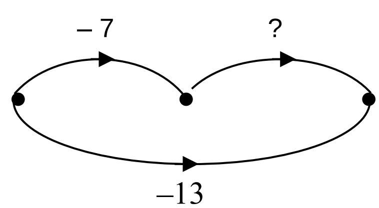
Techniques for performing operations, relationships between operations and the properties of operations only have real meaning when they are used to mathematize situations in order to solve problems. *Additive structures* deal with addition and subtraction, regardless of the type of numbers involved. It is essential that a variety of situations be presented: change (adding or taking away), uniting, comparing (more or fewer than) and combination of changes (positive, negative or mixed).

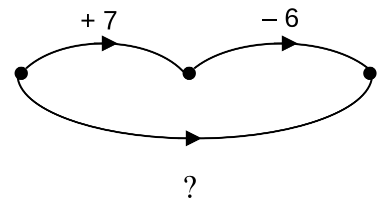
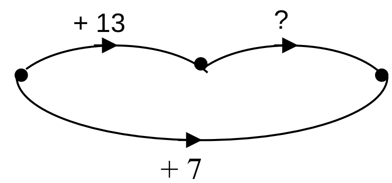
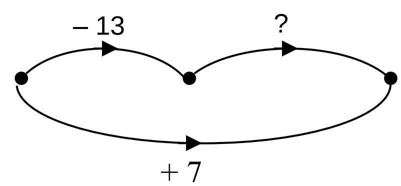
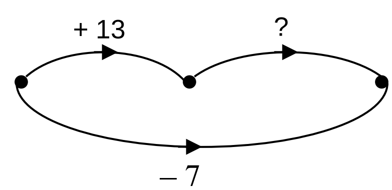
It is not necessary for students to know or memorize the different names for these structures. They must instead develop their own ways of representing them.

Structure or meaning	Situation[4]	**A Model** **(Students create their own representations according to the situation)**	Equation
Change (adding) Determine the final state	Gus had 7 objects. Melanie gave him 6 more. How many objects does Gus now have?		7 + 6 = ❒
Change (taking away) Determine the final state	Gus had 13 objects. He gave 6 of them to Melanie. How many objects does Gus have now?		13 – 6 = ❒
Change (adding) Determine the change involved	Gus had 7 objects. Melanie gave him some objects. Gus now has 13 objects. How many objects did Melanie give Gus?		7 + ❒ = 13
Change (taking away) Determine the change involved	Gus had 13 objects. He gave some to Melanie. Gus now has 7 objects. How many objects did Gus give Melanie?		13 – ❒ = 7

[4] The following examples each contain only two values. Teachers should, however, make sure to present situations consisting of several values and involving more than one meaning as well as superfluous or missing data.

Change (adding) Determine the initial state	Gus had some objects. Melanie gave him 6 more. Gus now has 13 objects. How many objects did Gus start with?		❒ + 6 = 13
Change (taking away) Determine the initial state	Gus had a certain number of objects. He gave 6 to Melanie. He now has 7 objects. How many objects did Gus start with?		❒ – 6 = 7
Uniting Determine the set	Gus has 7 objects. Melanie has 6. How many objects do they have in all?		7 + 6 = ❒
Uniting Determine a subset (complement)	Melanie and Gus have 13 objects all together. Gus has 7. How many objects does Melanie have?		7 + ❒ = 13 13 – 7 = ❒
Comparing (“more than”) Determine the comparison	Gus has 7 objects. Melanie has 6. How many more objects does Gus have than Melanie has?		7 = 6 + ❒ 7 – ❒ = 6
Comparing (“fewer than”) Determine the comparison	Gus has 7 objects. Melanie has 6. How many fewer objects does Melanie have than Gus has?		7 = 6 + ❒ 7 – ❒ = 6
Comparing (“more than”) Determine a set	Gus has 7 objects. He has 1 more object than Melanie. How many objects does Melanie have?		7 – 1 = ❒ 7 = ❒ + 1

Comparing (“fewer than”) Determine a set	Gus has 7 objects. Melanie has 1 fewer object than Gus. How many objects does Melanie have?		7 – 1 = ❒ 7 = ❒ + 1
Combination of changes (positive) Determine the gain	Yesterday, Gus received 7 objects. Today, he received 6 more. How many objects has he received in 2 days?		7 + 6 = ❒
Combination of changes (positive) Determine the change involved	Yesterday, Gus received 7 objects. Today, he received more but we don’t know how many. Knowing that he received 13 objects in the past 2 days, does he have more or fewer objects today? How many?		7 + ❒ = 13
Combination of changes (negative) Determine the loss	Yesterday, Gus gave away 7 objects. Today, he gave away 6. How many objects has he given away in the past 2 days?		7 + 6 = ❒
Combination of changes (negative) Determine the change involved	Yesterday, Gus gave away 7 objects. Today, he gave away some more, but we don’t know how many. Knowing that he has given away 13 objects in the past 2 days, how many objects has he given away today?		7 + ❒ = 13

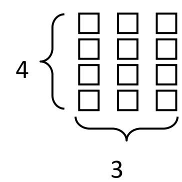
Combination of changes (mixed)[5] Determine the gain or loss	Gus had a certain number of objects. Yesterday, he received 7 objects. Today, he gave away 6. How many more or fewer objects does he have after 2 days?		7 – 6 = ❒
Combination of changes (mixed) Determine the change involved	Gus had a certain number of objects. Yesterday, he received 13. Knowing that at the end of 2 days, Gus has 7 more objects than he did at first, how many objects did he receive or give away today?		13 – ❒ = 7
Combination of changes (mixed) Determine the change involved	Gus had a certain number of objects. Yesterday, he gave away 13. Knowing that at the end of 2 days, Gus has 7 more objects than he did initially, how many objects did he receive or give away today?		–13 + ❒ = 7
Combination of changes (mixed) Determine the change involved	Gus had a certain number of objects. Yesterday, he received 13 objects. Given that at the end of the 2 days, Gus has 7 fewer objects than he did initially, how many objects has he received or given away today?		13 – ❒ = –7

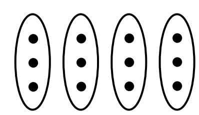
[5] Problems involving a mixed combination of changes require the use of integers. In Elementary Cycle Three, these problems are solved using a diagram or a number line.

MULTIPLICATIVE STRUCTURES

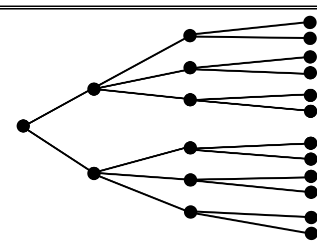
Techniques for performing operations, relationships between operations and the properties of operations only have real meaning when they are used to mathematize situations in order to solve problems. *Multiplicative structures* deal with multiplication and division, regardless of the types of numbers involved. It is much more important to present students with a variety of situations than to emphasize the different names associated with the structures such as: repeated addition, combination or Cartesian product, rectangular arrangement, area and volume, comparison (times as many), repeated subtraction, sharing, number of times x goes into y, and comparison (times fewer than). The table below presents a variety of situations involving varying levels of difficulty.

<table>
<tr><th colspan="2">Structure or meaning</th><th>Situation[6]</th><th>A Model
(Students create their own representations according to the situation)</th><th>Equation</th></tr>
<tr><td colspan="2">Rectangular arrangement</td><td>In the classroom, there are 3 rows containing 4 desks each.

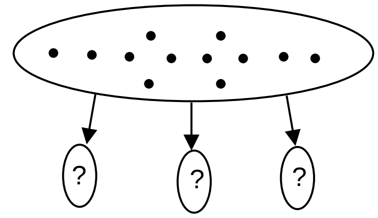
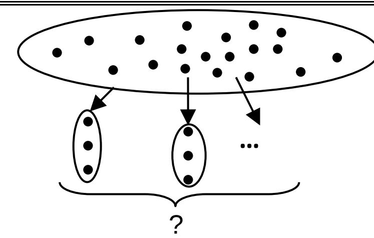
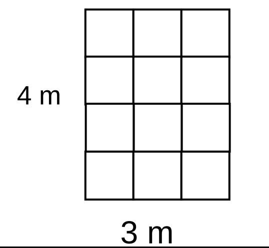
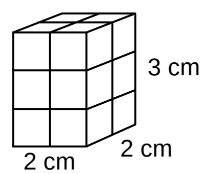
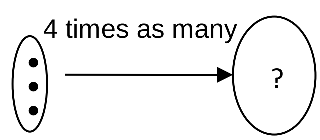
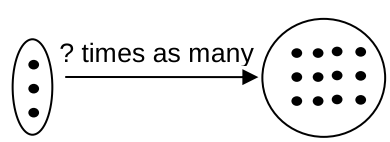
How many desks are in the classroom?</td><td>
</td><td>3 × 4 = ❐
or 4 x 3 = ❐</td></tr>
<tr><td colspan="2">Repeated addition</td><td>Gus receives 3 objects per day.

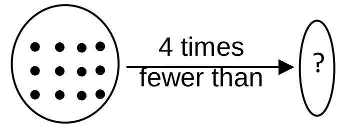
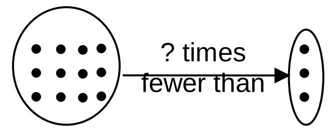
How many objects will he receive in 4 days?</td><td></td><td>3 + 3 + 3 + 3 = ❐

3 × 4 = ❐
or 4 × 3 = ❐</td></tr>
<tr><td rowspan="2">Combination</td><td>Cartesian product</td><td>Gus has 4 shirts and 3 pairs of pants.

How many combinations of pants and shirts can he wear?</td><td><table><tr><th></th><th>C1</th><th>C2</th><th>C3</th><th>C4</th></tr><tr><td>P1</td><td>P1C1</td><td>P1C2</td><td>P1C3</td><td>P1C4</td></tr><tr><td>P2</td><td>P2C1</td><td>P2C2</td><td>P2C3</td><td>P2C4</td></tr><tr><td>P3</td><td>P3C1</td><td>P3C2</td><td>P3C3</td><td>P3C4</td></tr></table></td><td>4 × 3 = ❐
or 3 × 4 = ❐</td></tr>
<tr><td>Tree</td><td>At the cafeteria, there are 2 types of soup, 3 main dishes and 2 desserts on the menu. How many different meals can you have?</td><td></td><td>2 × 3 × 2 = ❐</td></tr>
</table>

6 The following examples generally contain only two values each. Teachers should make sure to present situations consisting of several values and involving more than one meaning as well as superfluous or missing data.

Sharing	There are 12 objects in a bag. They are distributed evenly among 3 friends. How many objects does each friend receive?		12 ÷ 3 = ❒
Number of times x goes into y	12 objects must be placed in bags. Each bag can hold 3 objects. How many bags are needed?		12 ÷ 3 = ❒
Area	A flower bed is 4 m wide and 3 m long. What is the area of this flower bed?		4 × 3 = ❒ or 3 × 4 = ❒
Volume	A box in the shape of a rectangular prism is 2 cm wide, 2 cm deep and 3 cm high. What is the volume of the box?		2 × 2 × 3 = ❒ 2 × 3 × 2 = ❒ or …
Comparison (“times as many”) Determine one of the subsets	Gus has 3 objects. Melanie has 4 times as many objects. How many objects does Melanie have?		3 x 4 = ❒
Comparison (“times as many”) Determine the relationship	Gus has 3 objects and Melanie has 12. How many times more objects does Melanie have than Gus?		3 x ❒ = 12 or 12 ÷ 3 = ❒

Comparison ("times fewer than") Determine one of the subsets	Gus has 12 objects. This is 4 times as many objects as Melanie has. How many objects does Melanie have?		12 ÷ 4 = ❒
Comparison ("times fewer than") Determine the relationship	Gus has 12 objects and Melanie has 3. How many times fewer objects does Melanie have than Gus?		12 ÷❒ = 3 or 12 ÷ 3 = ❒

OPERATIONS INVOLVING NUMBERS

Individual and conventional processes

With respect to processes, the students spontaneously devise their own ways of doing things by using instruments or technology, and explore these methods in order to understand how they work. For example, instead of using recognized algorithms, the students can begin by carrying out arithmetic operations based on relatively unstructured intuition. Measurements can be taken using any object as a unit of measurement. (*Québec Education Program*, p. 144)

However, mathematics has its own processes and instruments that have become well- established conventions over time. In addition, when it comes to learning about instruments, the instructional goal should be to ensure that the students are able to use these conventional tools intelligently and to understand what they are doing, while developing their measurement sense. (*Québec Education Program*, p. 144)

As students gradually develop their number sense and operations sense, they will be called on to develop their own processes and adopt conventional ones. In order for students to understand conventional algorithms better, they must develop their own computation techniques. This step must not be skipped, although some parents and teachers find it hard to admit that there are other methods that are just as effective as conventional algorithms.

To calculate correctly, students use various concrete, semi-concrete and symbolic means to develop their number sense and their operations sense. At first, students simulate a problem by imitating it, simulating it using drawings, physical objects or manipulatives (fingers, tokens, blocks, etc.). They recite counting rhymes, count units, group units, take groupings apart, move objects around, add, take away, bring together, share and draw their mathematical activities.

Next, students are guided to explore invented solutions and their own processes. ―We will refer to any strategy other than the traditional algorithm and that does not involve the use of physical materials or counting by ones as an *invented strategy*. These invented strategies might also be called *personal and flexible strategies.*||[7] These personal processes are not restricted to making computation easier; their development and regular use have many benefits:

- Base-ten concepts are enhanced||[8]: place

[8] John A. Van de Walle and LouAnn H. Lovin, *Teaching Student-Centered Mathematics* (Allyn & Bacon, 2006), Volume 1: Grades K-3, 159

value and breaking down of numbers.

- Students make fewer errors||[9] because they are using methods that they understand.
- Less reteaching is required. Students rarely use an invented strategy that they do not understand. The supporting ideas are firmly networked with a sense of number, thus making the strategies more permanent. In contrast, students are frequently seen using traditional algorithms without being able to explain why they work (Carroll & Porter, 1997).
- Invented strategies provide the basis for mental computation and estimation.||[10]
- Flexible, invented strategies are often faster than the traditional algorithms.||[11]

Unlike conventional computation processes, personal computation processes are based on the numbers rather than on the single digits that make up the number. This has the advantage of creating links to understanding the base-ten numerical system. In addition, by favouring computation that starts on the left side, personal processes enable the answer's order of magnitude to be determined rapidly. Last, personal processes are flexible and fluid, enabling several strategies to be used to perform calculations and to be adapted to the numbers involved in the calculation. Although these personal processes, generated by the student's understanding and development, can vary widely from one student to another, they are all based on the relationships between numbers and operations.

In class, the teacher facilitates discussions about personal computation processes in mathematics. These discussions result in the sharing of personal processes and encourage the discovery of affinities among the various ways the students proceed. The personal processes of all students are enriched and become more effective when they make these links. The teacher guides the discussion by referring to personal processes that students have used to develop their understanding of mathematical concepts and by orienting student learning toward effective methods. In this way and as time goes on, their understanding of processes will enable students to choose the better model for solving a particular problem. Furthermore, in-class discussion of personal processes will give students with difficulties the opportunity to fully understand and adopt the effective processes of others.

The teacher's role is to help students establish clear and complete lines of reasoning, to encourage them to talk about what they do, and to help them make permanent connections thus enabling them to understand the concepts associated with the actions performed during the operation. Teachers ask students to model a process or model it themselves, being sure to explain the reasoning behind it. The teacher must present a variety of problem-solving situations. This variety will allow students to develop a larger stock of models, expand their skills, take advantage of connections and develop flexibility in using operations.

Next, students will learn conventional processes. It is important that students who use a conventional process, like any other process, understand how it works and are able to explain it. A gradual transition from manipulatives to the conventional operating processes will help students to understand the latter. The conventional algorithm, once correctly understood, will become one more method for students to add to their computation repertoire.

[9] Ibid. 160

[10] Ibid. 160

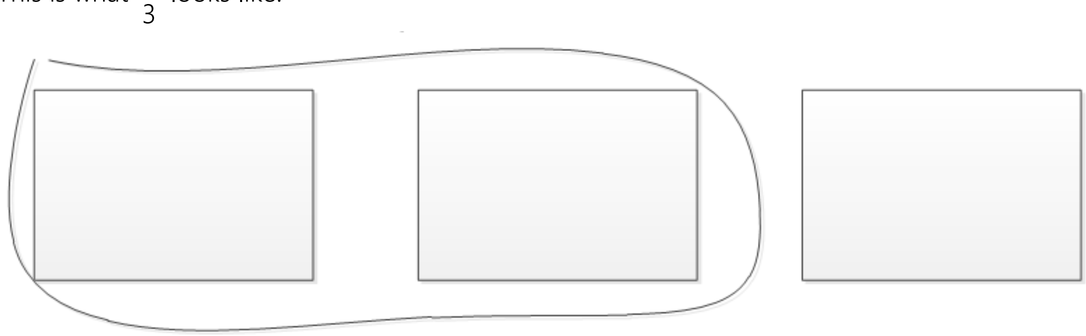
[11] Ibid. 161

Some examples of computation processes

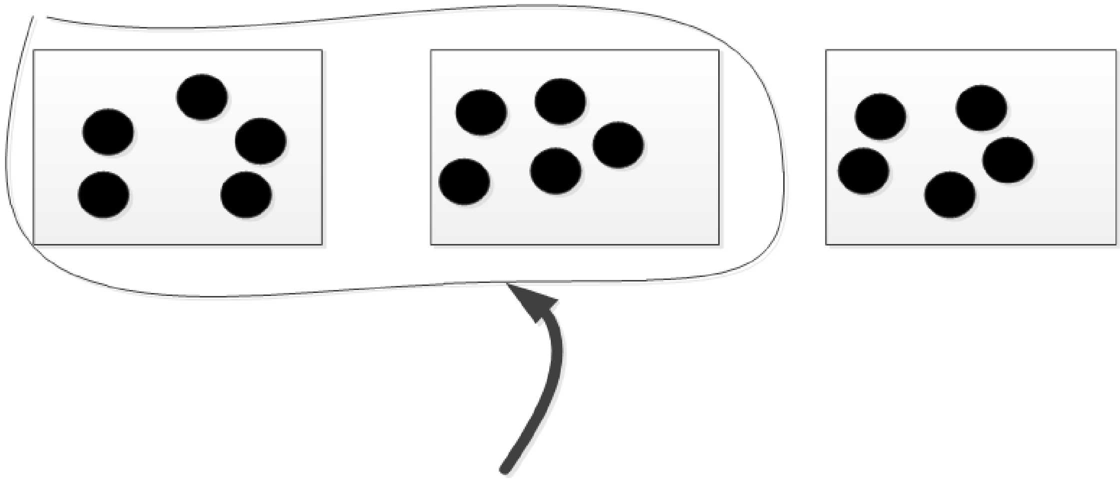
Marie is giving away 2/3 of her 15 marbles. How many marbles is she going to give away?

- Personal process

1 – This is what $\frac{2}{3}$ looks like.



2 – I distribute my 15 marbles among my three areas to see how many marbles that represents.



3 – There, $\frac{2}{3}$ of the marbles makes 10 marbles. Marie is going to give away 10 marbles.

- Conventional process

1. 2/3 of 15 marbles
2. I take 15 and divide them into 3 parts   15 ÷ 3 = 5
3. I take 2 of the parts   5 X 2 = 10
4. Marie will give away 10 marbles.

- Conventional process

2/3 of 15 marbles

15 X 2 ÷ 3 = 10, Marie will give away 10 marbles.

Mental computation

By the end of Cycle Three, the students mobilize their own processes as well as conventional processes to do mental and written computations involving the four operations with natural numbers and decimals. (*Québec Education Program*, p. 147)

Mental computation is the process of performing computations with no, or practically no, use of pen and paper or a calculator. Mental computation is not applying an algorithm in one's head but using flexible and varied efficient computation processes. Mental computation techniques are used to make an approximation (estimate of order of magnitude) or to determine an exact result.

Mental computation skills can be developed in various ways in learning situations—through games or in guided or autonomous learning situations. The teacher asks students to talk about their mental computation strategies in order to discover the similarities and differences between the strategies, as well as their strengths and limitations. The teacher helps the students to make connections and to adopt other mental computation strategies.

Number sense is crucial in this context. In particular, students must feel comfortable working with the concepts of place value and number decomposition, finding the order of magnitude and switching from one way of writing numbers to another. Operations sense is also developed as students learn to use the relationships between operations and the properties of operations effectively. These computation processes often result from the transference of models used in learning the meaning of operations.

The following table presents a few examples of mental computation strategies.

Process	Example	Knowledge
Adding by making the first term a multiple of 10	47 + 14 = 47 + (3 + 11) = (47 + 3) + 11 = 50 + 11 = 61	Place value Decomposition Associative law
Addng by making the first term a multiple of 10 Compensation	37 +16 = 37 + 3 + 16 – 3 = 40 + 16 – 3 = 56 – 3 = 53	Associative law
Adding the tens, then adding the ones	49 +28 = 40 + 9 + 20 + 8 = 40 + 20 + 9 + 8 = 60 + 17 = 77	Place value Decomposition Commutative law
Subtracting the tens and then the ones	46 – 12 = 46 – 10 – 2 = (46 – 10) – 2 = 36 – 2 = 34	Place value Decomposition
Subtracting by making the second term a multiple of 10	54 – 18 = 54 – 20 + 2 = (54 – 20) + 2 = 34 + 2 = 36	Place value Decomposition

Subtracting by first making sure that each term has the same number of units Compensation	$51 - 38 = 51 + 7 - 38 - 7$ $= (58 - 38) - 7$ $= 20 - 7$ $= 13$	Decomposition
Multiplying by decomposing the multiplicand (1st factor)	$23 \times 4 = (20 + 3) \times 4$ $= (20 \times 4) + (3 \times 4)$ $= 80 + 12$ $= 92$	Decomposition Distributive law
Multiplying by decomposing the multiplier (2nd factor)	$23 \times 12 = 23 \times (10 + 2)$ $= (23 \times 10) + (23 \times 2)$ $= 230 + 46$ $= 276$	Decomposition Distributive law
To multiply by 4 or 8, multiply by 2 twice or three times	$13 \times 4 = 13 \times 2 \times 2$ $= 26 \times 2$ $= 52$	Decomposition (into prime factors)
To multiply by 6, multiply by 2, then multiply by 3 or vice versa	$15 \times 6 = 15 \times 2 \times 3$ $= 30 \times 3$ $= 90$	Decomposition (into prime factors)
To multiply by 5, multiply by 10, then divide by 2 or vice versa	$28 \times 5 = 28 \times 10 \div 2$ $= (28 \times 10) \div 2$ $= 20 \div 2$ $= 140$	$28 \times 5 = 28 \div 2 \times 10$ $= (28 \div 2) \times 10$ $= 14 \times 10$ $= 140$
Dividing by making a multiple of the divisor appear in the dividend	$42 \div 3 = (30 + 12) \div 3$ $= (30 \div 3) + (12 \div 3)$ $= 10 + 4$ $= 14$	Decomposition Distributive law
Dividing by making a multiple of the divisor appear in the dividend	$54 \div 3 = (60 - 6) \div 3$ $= (60 \div 3) - (6 \div 3)$ $= 20 - 2$ $= 18$	Decomposition Distributive law
Dividing by breaking down the divisor into several factors	$54 \div 18 = (54 \div 2) \div 9$ $= 27 \div 9$ $= 3$	Decomposition into several factors
To divide by 5, multiply by 2, then divide by 10 or vice versa	$140 \div 5 = 140 \times 2 \div 10$ $= 280 \div 10$ $= 28$	$140 \div 5 = 140 \div 10 \times 2$ $= 14 \times 2$ $= 28$
Compensation	$80 \times 0.3 = (80 \div 10) \times (0.3 \times 10)$ $= 8 \times 3$ $= 24$	
Multiples of the same number Compensation	$3500 \div 500 = (3500 \div 100) \div (500 \div 100)$ $= 35 \div 5$ $= 7$	

Using technology

Technology can prove to be a valuable tool that will help the students solve situational problems, understand concepts and processes and carry out assigned tasks more efficiently. (*Québec Education Program*, p. 141)

With respect to processes, the students spontaneously devise their own ways of doing things by using instruments or technology, and explore these methods in order to understand how they work. (*Québec Education Program*, p. 144)

In all cycles, calculators may be used to good advantage as a calculation, verification and learning tool (e.g. in situations involving patterns, number decomposition, or the order of operations). (*Progression of Learning*, p. 11)

It is important to allow students to use technological tools like computers and calculators because they are a big part of daily life. Often, though, young people associate using a calculator with making it ―easier‖ to perform a computation and do not understand that calculators only provide the answer to a mathematical operation and not the solution to a problem. It is therefore essential to teach students to use calculators wisely.

Calculators are useful for solving situational problems that have several steps because students have to understand the problem and decide which computations to perform, with the emphasis being placed on reasoning, research and the implementation of strategies that call on knowledge, rather than on the computational process. Calculators can also be useful for verifying operations or carrying out operations on numbers that go beyond the requirements of the program.

In order to make good use of calculators, students must interpret the problem's data and enter it correctly, which means they have to have a good grasp of the tool's functions. They must also be able to correctly interpret the numbers that appear in the graphic display, especially decimals when they are the results of a division (e.g. students may interpret 7.5 as 7 with a remainder of 5 instead of 7 and five tenths).

The computer is useful for:

- applying various strategies for solving problems (drawing, spreadsheet and simulation software)
- publishing information related to the solution (word processing, drawing and spreadsheet software)
- researching data
- creating graphic representations of data (spreadsheet)
- producing drawings of solids, plane figures, frieze patterns, tessellations (drawing software)
- simulating random experiences
- consulting Web sites to do with mathematics, lexicons and databases
- participating in activities on interactive mathematics Web sites

MANIPULATION AND MANIPULATIVES

The students primarily use manipulative materials. (*Québec Education Program*, p. 144)

As with everything they study, students will find it that much easier and more rewarding to learn about mathematical concepts and processes if learning situations are made concrete or accessible. (*Québec Education Program*, p. 145)

Several authors claim that children become better acquainted with concepts by ―handling‖ them. Physical images lead to mental images and concrete actions lead to mental operations. By starting with manipulation, students carry out their mathematical activities better. Thus, manipulation, when it is properly understood and used, supports the internalization of mathematical concepts and processes. This is why authors are in agreement that manipulation is never a waste of time and that it has to be accepted that students can go back to it as needed without it being considered as regression.

But, be careful! Manipulatives are not synonymous with concepts. A model (an object, image or drawing) can help in visualizing relationships supposed by a concept and in talking about it, but it is not the concept itself. Van de Walle and Lovin refer to Thompson's 1994 statement that it is incorrect to claim that a model ―illustrates‖ a concept. To illustrate is to show. That would mean that, in looking at the model, one would see an example of the concept. In fact, all you see is the physical object; it is only your mind that can associate the mathematical relationship with the object. They go on to claim that, for a person who does not know this relationship, there is nothing that connects the model to the concept. Mathematical concepts are relationships built in our minds.[12][12] The danger lies in students using the manipulatives in a mechanical fashion without making the connections with the mathematical concepts. The teacher must ensure that there is real learning and not just a ―mindless‖ use of the manipulatives. Van de Walle and Lovin argue that, in general, mathematical models and tools play a similar role, in other words, they serve as a test platform for emerging ideas. That makes these tools, in a way, ‗toys for thinking,' ‗toys for experiencing' and ‗toys for expressing oneself,' It is difficult for of any age to present and check abstract relationships just using words. The models provide them with materials for thought, for exploration, for reasoning and for expression[13].

The Québec Education Program recommends concrete or semiconcrete manipulatives in all the branches of mathematics and all education cycles.

Concrete manipulatives consist of objects (physical collections or a group of materials). They can be classified according to their nature:

- unstructured (e.g. tokens, marbles, stacking cubes)
- structured (e.g. Cuisenaire rods, base-ten manipulatives)
- symbolic (e.g. abacus, meter, scoreboard, coins)

Semi-concrete manipulatives consist of illustrations, diagrams, drawings, number charts and number lines.

[12] John A. Van de Walle and LouAnn H. Lovin, *Teaching Student-Centered Mathematics* (Allyn & Bacon, 2006), Volume 3: Grades 5-8.

[13] Ibid

Some examples of occasions for using manipulatives

- Count or recite a counting rhyme of natural numbers with the help of a numbers chart.
- Enumerate groupings—physical or drawn.
- Represent natural numbers in various ways or associate a number with a group of objects or with drawings.

  - Emphasis on groupings using *apparent, accessible objects or drawings:* —This manipulative is the easiest for young students who are beginning to learn about numbers. With this type of manipulative, related units are in the first grouping; students see the 10 units that make up a group of ten and these units are accessible, in other words, students can go and find 10 units directly in the group of ten. The same goes for all the groupings and so one sees the elements that compose the second grouping. This type of material is extremely useful for groupings and their composition||[14][14](e.g. tokens, nesting cubes).

  - Emphasis on exchanging by using *apparent, nonaccessible groupings:* "With this type of manipulative material, the units are not directly accessible. Students cannot directly withdraw the 10 units that compose the group of 10. However, this type of manipulative lets one see the units that compose the first grouping and the elements that compose the other groupings. This type of manipulative enables one to see another characteristic of our number system: exchange. It is extremely useful for groupings and their composition.||[15][15] (e.g. base-ten manipulatives).

  - Emphasis on place value in nonapparent, nonaccessible groupings, using materials for which groupings are symbolic*:* —With this type of manipulative material, the groupings are neither apparent nor accessible, they are symbolic. Students therefore must remember the rule of grouping (in our number system, this is a standard rule: one works with base-ten with regular groupings). As it is more difficult to work with these manipulatives, they should be introduced after the other two||[16][16] (e.g. abacus, meter, coins).

- Validate the equivalence of two fractions with the help of self-stacking boxes or illustrations.
- Represent decimals using base-ten manipulatives or illustrations.
- Translate a situation using concrete manipulatives, diagrams or equations and vice versa (using various meanings of multiplication and division).
- Use personal processes to develop written computation processes (addition and subtraction), by using concrete manipulatives or drawings.
- Compare and construct solids. —Manipulating images on the computer seems to make learning geometry easier because this device allows certain activities to be carried out that would be difficult to do otherwise.||[17][17]
- Construct rulers.
- Manipulate conventional units of measure.
- Experiment with activities related to chance by using diverse manipulatives (e.g. wheels, rectangular prisms, glasses, marbles, thumbtacks, and dice with 6, 8 or 12 faces).

It is important to remember that manipulation is not a waste of time and that it is present in all branches of mathematics until the end of Elementary Cycle Three.

[14] Louise Poirier, *Enseigner les maths au primaire, notes didactiques* (Saint-Laurent: ERPI, 2000), 38 [*translation*].
[15] Ibid., 39 [*translation*].
[16] Ibid., 39 [*translation*].
[17] Ibid., 135 [*translation*].

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