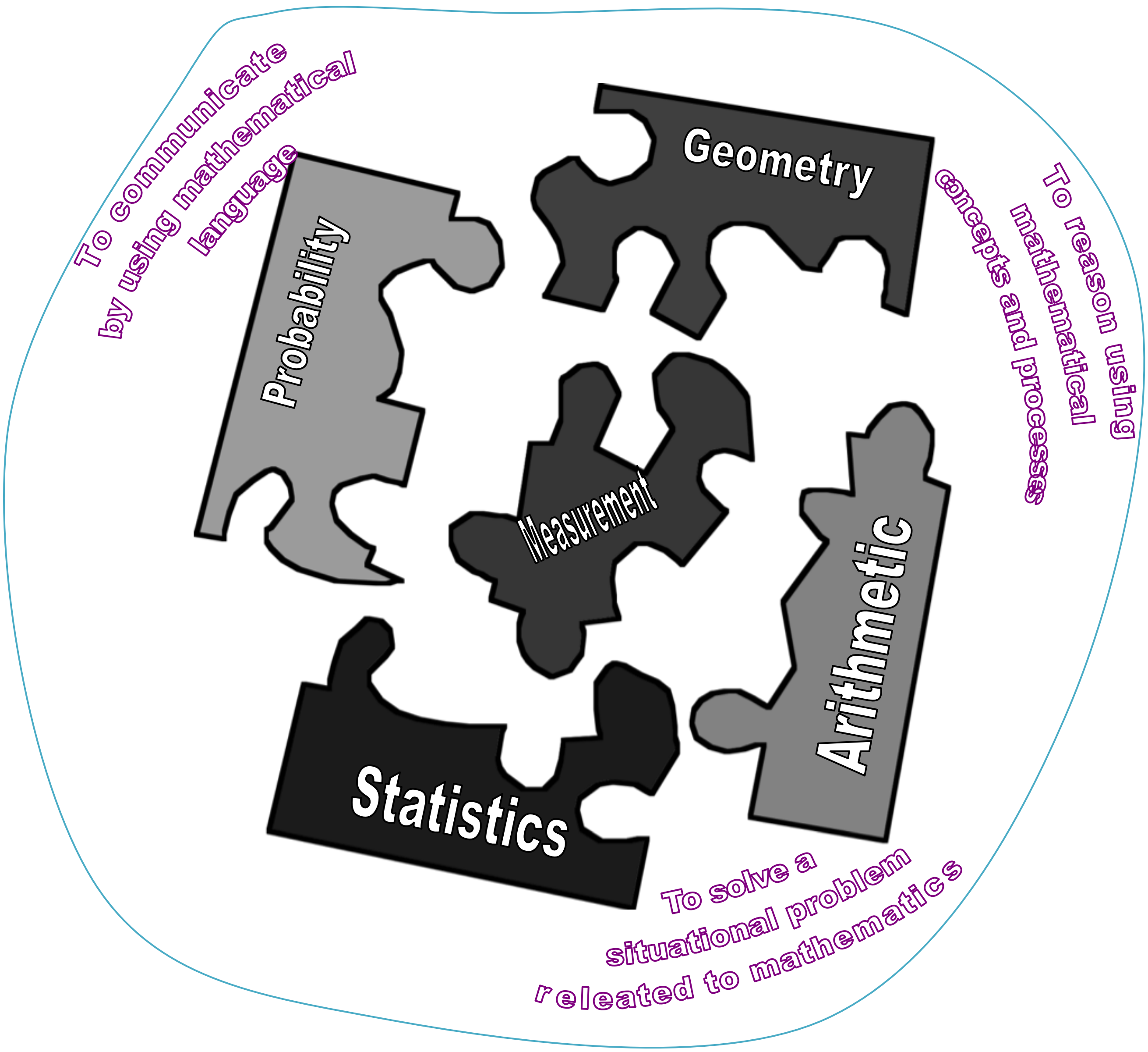
Progression of Learning

Mathematics

Companion Document



Working Document
Fall 2009

Progression of Learning
Mathematics

This document contains passages (in italics) from the document *Progression of Learning - Mathematics* as well as suggestions, specifications and information related to:

- networks of concepts and processes
- elements of mathematical language
- multiplicative and additive structures
- students' own processes (mental and written computation)
- geometry and measurement

About mathematics . . .

Mathematics is a science and language that involves abstract objects[1]. For example, although a perfect square does not actually exist in nature, we can see visual representations of it. Students develop their mathematical thinking through personal experiences and interaction with peers. This learning is based on concrete situations that are often related to everyday life. To foster learning, the teacher proposes different activities and situations with varying levels of complexity, requiring students to manipulate various concrete materials. These situations enable students to question their representations, and to build on and enrich the understanding they already have. This leads students to develop their own networks of mathematical concepts and processes. An understanding of numbers, operations, space and data derived from random experiments or statistical reports is developed by working with concrete situations or objects to gradually gain an understanding of abstract mathematical concepts. Throughout their elementary school education, the teacher supports students as they alternate between using concrete objects and abstractions. It is therefore important to allow students time to develop their understanding so that they can give structure to their mathematical learning in various contexts.

In all branches of mathematics, students should deal with situations that require explanations or answers to questions such as "Why?", "Is this always true? Is there an example that contradicts this statement?" and "What happens when..." These questions encourage students to reason, acquire mathematical knowledge, interact and explain their approach.

In this way, students build a set of tools that will allow them to communicate[2] appropriately using mathematical language, reason effectively by making connections between mathematical concepts and processes, and solve situational problems. By using mathematical concepts and various strategies, students can make informed decisions in all areas of life. Combined with learning activities, the situations experienced by students promote the development of mathematical skills and attitudes that allow them to mobilize, consolidate and broaden their mathematical knowledge.

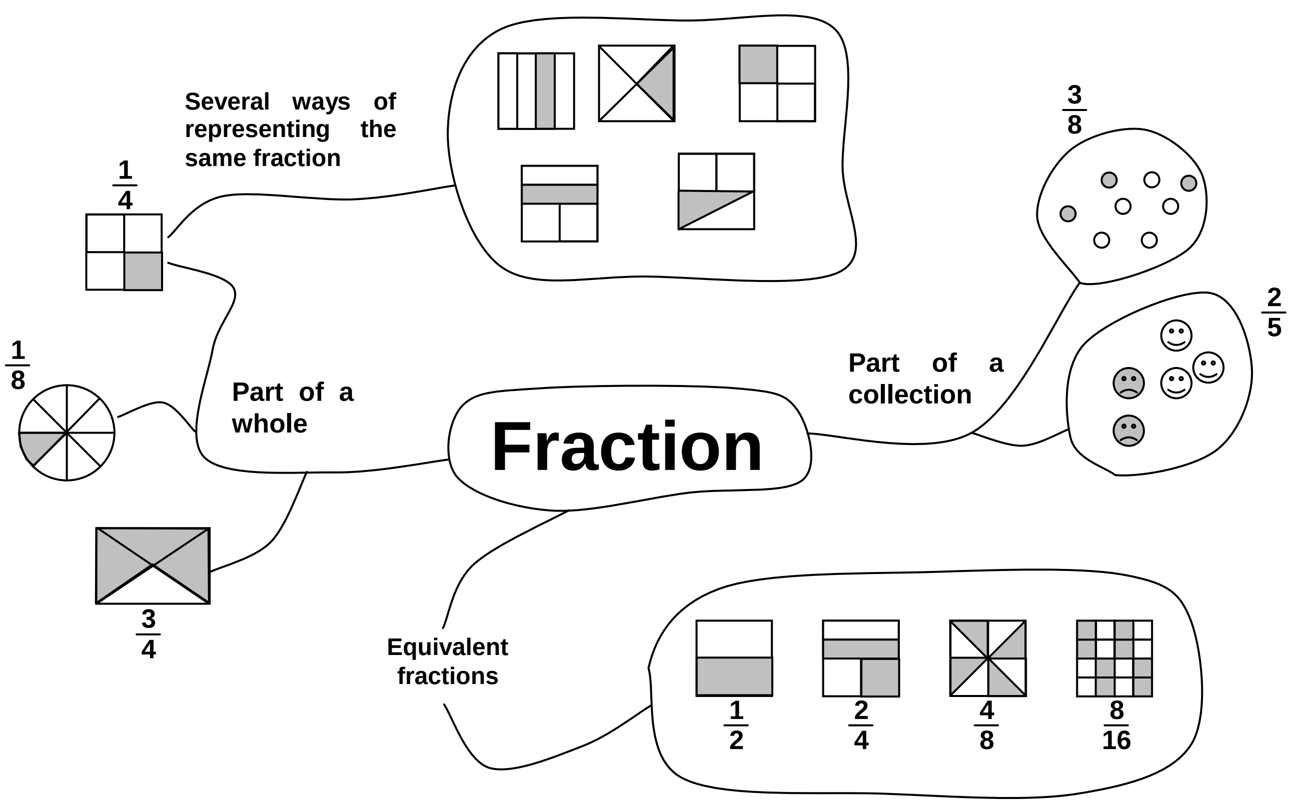
[1] For example, although a perfect square does not actually exist in nature, we can see visual representations of it.
[2] Taking the cross-curricular competency *To communicate appropriately* one step further, the competency *To communicate by using mathematical language* requires that students learn and coordinate the elements of mathematical language (types of representation) that are used in conceptualizing mathematical objects. When using these mathematical concepts and processes, students must interpret or produce mathematical messages.

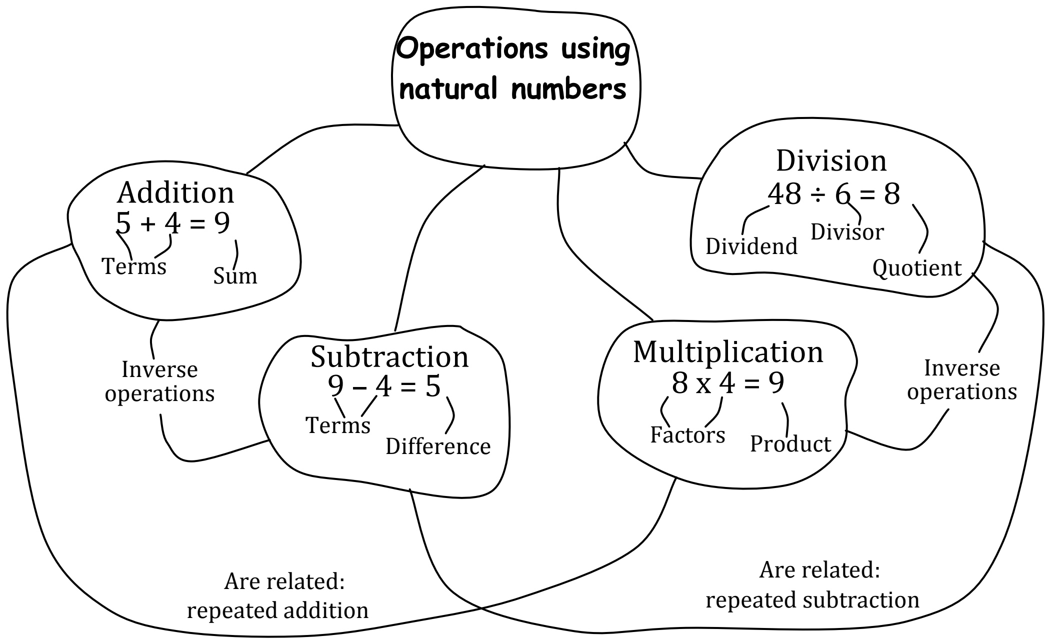
The table below provides indicators of whether students have learned and mastered a concept, a process or a strategy.

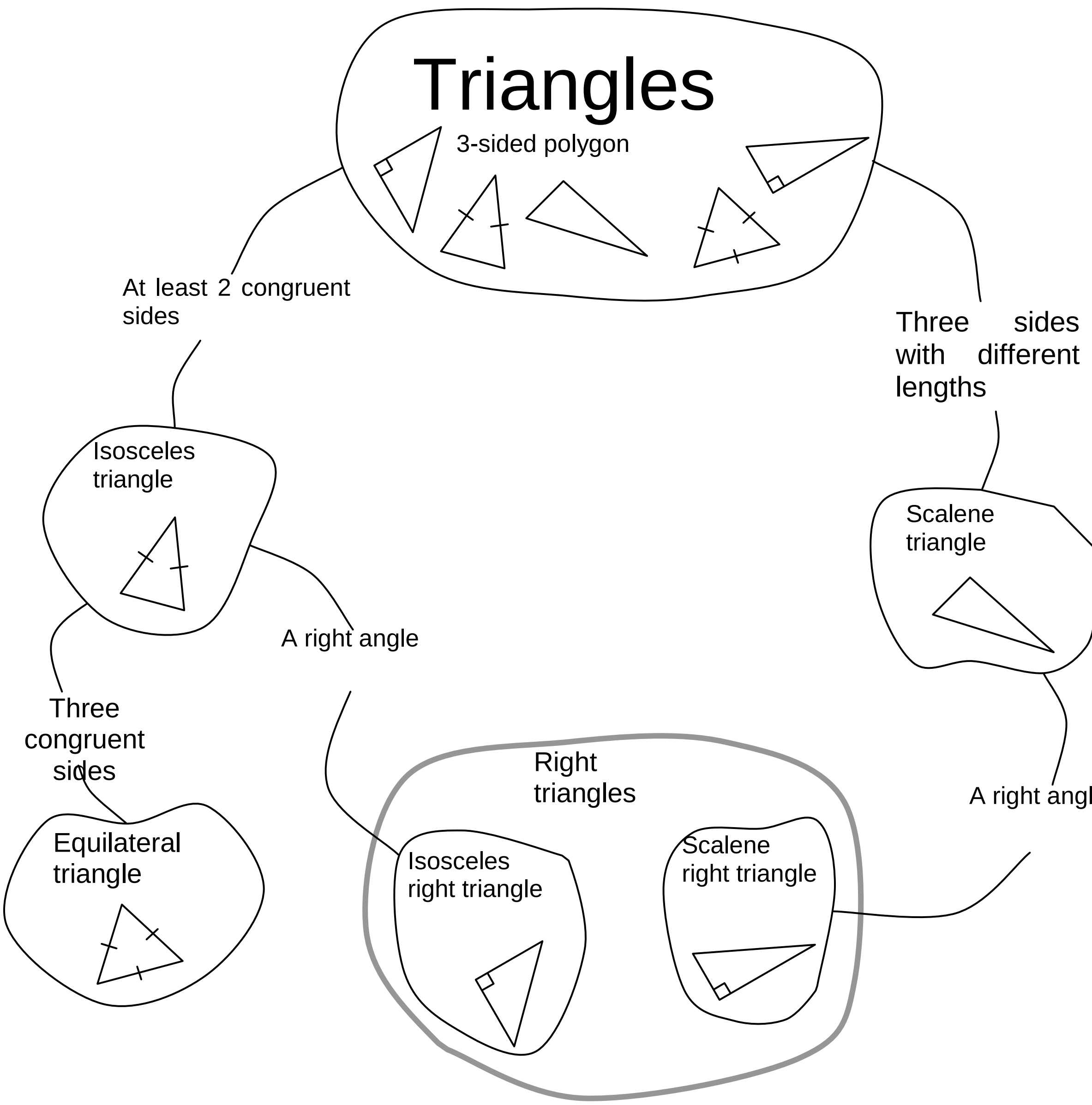
I have learned a concept if . . .	I have learned a process or a strategy if . . .
• I can identify its essential qualities (properties). • I can produce examples and counterexamples. • I can develop a personal definition. • I can relate the concept to other concepts. • I can recognize the concept in a situation.	• I can produce a description, definition or example of the process. • I can understand its importance and purpose. • I know how to implement it. • I know and can explain all the steps involved in implementing it. • I can compare my process or strategy with other processes or strategies. • I can use concepts and properties to explain the steps in my process or strategy. • I know when to use it.

NETWORKS OF CONCEPTS AND PROCESSES

Here are three examples of how students could create networks of concepts and processes outlined in the program. Each network is dynamic and individual, and evolves as the student progresses.

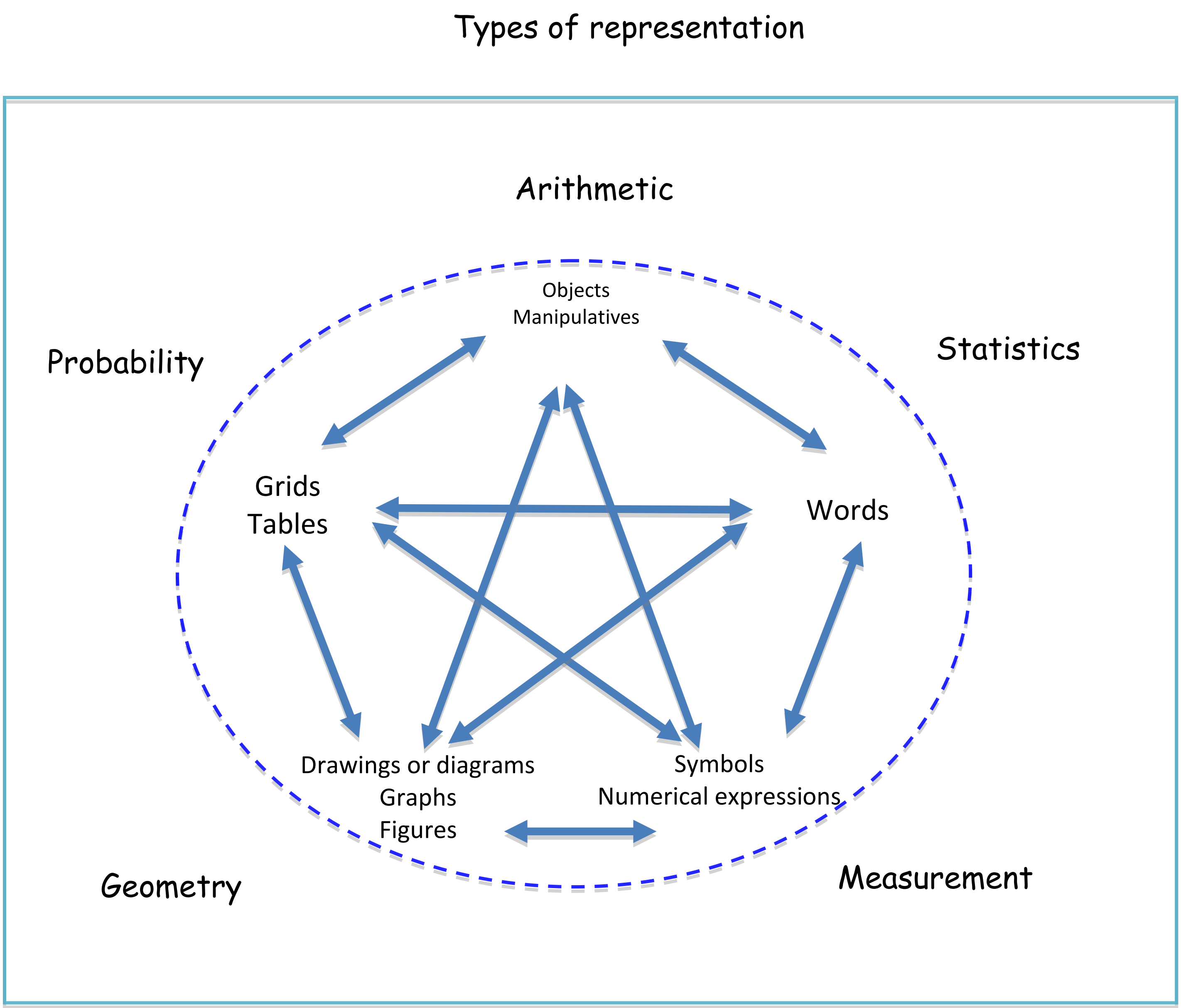



Operations using natural numbers
Addition
5 + 4 = 9
Terms
Sum
Subtraction
9 – 4 = 5
Terms
Difference
Division
48 ÷ 6 = 8
Dividend
Divisor
Quotient
Multiplication
8 x 4 = 9
Factors
Product
Inverse operations
Inverse operations
Are related: repeated addition
Are related: repeated subtraction


Triangles
3-sided polygon
At least 2 congruent sides
Three sides with different lengths
Isosceles triangle
Scalene triangle
A right angle
Three congruent sides
A right angle
Right triangles
Equilateral triangle
Isosceles right triangle
Scalene right triangle

ELEMENTS OF MATHEMATICAL LANGUAGE

Mathematical language is complex[3] because it is composed of different languages, including everyday language. In elementary school, students become familiar with and master the elements of this language, which is made up of words, tables, objects, figures, graphs and symbols. It is important that students learn to choose one or more types of representation appropriate to the situation to identify the information conveyed through different types of representation and to follow the rules and conventions for writing mathematical information. Learning the elements of mathematical language and how to coordinate them involves making connections between the types of representation in every branch of mathematics.



[3] It should be noted that mathematical language, which is related to conceptualization, is based on conventions governed by precise rules. Even young students should be introduced to vocabulary and symbolism in such a way as to avoid inaccuracy and vagueness. A lack of precision could undermine students' comprehension and academic progress and prove difficult to correct.

MATHEMATICAL LANGUAGE

The following table presents some examples of details specific to mathematical language.

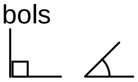
<table>
<tr><td colspan="2">Type of statements
• Statements consisting of only words
Example: True or false? If a rhombus has four right angles, it is a square.

• Statement consisting of both words and mathematical symbols
Example: What is the value of the expression (7 + 6) – 3 x 4?

• Statements consisting of only mathematical symbols
Example: 3 x 4 = 12</td></tr>
<tr><td>Types of symbols
• Symbols used to name objects
Examples: 8, $\frac{3}{5}$, ∠

• Symbols used in operations
Examples: +, –, ×, ÷

• Symbols used in relations
Examples: >, <, =, ≠, ⊥, ⫽

• Graphic symbols
Examples: </td><td>Meaning of symbols
• The order and position of symbols affects their meaning.
Examples:
34 and 43

$\frac{3}{5}$ and $\frac{5}{3}$

1,234 and 12.34 and 123.4

3^2 and 2^3</td></tr>
<tr><td>Rules for writing mathematical information
• Numbers: A space is required after each set of three digits. However, this space is not always required for a four-digit natural number.
Examples: 12 345
123 456
1234 or 1 234

• Ordered pairs: A space is required after the comma.
Example: (3, 2)

• A space is required between a number and a unit of measurement
Example: 3.5 g, 4 cm,</td><td>Types of notation
• Exponential: 3^2 reads “3 to the power of 2”

• Fractional: $\frac{2}{3}$ reads “two thirds”
$\frac{5}{8}$ reads “five eighths”

• Decimal: 2.35 reads “2 and 35 one hundredths”

• Percentage: 3% reads “3 percent”</td></tr>
<tr><td>Types of words and their meanings
• Terms specific to mathematics
Examples: polygon, parallelogram, perpendicular, random, fraction, hundreds, etc.

• Terms with a mathematical meaning different from their everyday meaning (polysemy)
Examples: product, factor, centre, volume, length, area, etc.</td><td>Reading symbols and expressions
• Phrase corresponding to one symbol
Examples:
=: . . . is equal to . . .
≥: . . . is greater than or equal to . . .

• Different expressions used in reading 12 – 5
Examples:
twelve minus five
twelve subtract five
five less than twelve
take five away from twelve
the difference between twelve and five</td></tr>
</table>

ARITHMETIC

Many concepts and processes to be acquired and mastered in arithmetic constitute the building blocks of mathematics and are more important, since they are applied in all other branches of this subject.

Understanding and writing numbers

Number sense is a concept that is developed in early childhood and is refined as students progress through school. In elementary school, number sense is initially developed as students learn about natural numbers and then enriched when they go on to study rational numbers.[4]

As students develop an understanding of the concepts of quantity and size they begin to grasp the usefulness of numbers for either expressing a code, a quantity or a size (cardinal aspect), or for ranking quantities and sizes (ordinal aspect). It is essential that students broaden their understanding of numbers through the study of operations, among other things.

At the outset, counting rhymes, counting, constructions, representations, ordering and establishing relationships among numbers are essential in order for students to understand number systems. Using appropriate manipulatives, students first learn about counting groups (grouping) and gradually replace this concept with place value. However, care must be taken not to progress too quickly from one concept to another, as this could affect the way students understand operations or learn new numbers.

It is in elementary school that students acquire the basic tools for understanding and using fractions. Students must first understand concepts (meaning) before they can understand calculation processes (operations). This can be achieved by allowing students to systematically use concrete materials and pictorial representations when dealing with situations involving fractions.

Exploring fractions also reinforces writing conventions, which students must learn to recognize, decode, and use correctly. A student has truly mastered the various ways of representing numbers when he or she can switch from one type of representation to another as needed in given situations. Students' ability to do this reveals that they have grasped these various representations and can recognize numbers written as fractions, decimals and percentages. This skill also confirms students' ability to better understand the elements of a task and choose one or more representations that are best suited to solving a problem. In addition, it is sometimes necessary to convert from one representation to another when using units of measurement (length, surface area, space, money, etc.). The skills developed in handling these conversions will be used in every branch of mathematics.

In Elementary Cycle Three, students' introduction to integers marks an important step in their mathematical learning. They are introduced to a new dimension of numbers, which were initially used to count (natural numbers), share out (fractions) and measure (decimals). The study of integers involves new concepts such as symmetrical numbers, opposite numbers, and positioning on a number line and in a Cartesian coordinate graph. Moreover, integers put an end to the notion that numbers only denote quantities–they are also mathematical entities whose behaviour is subject to rules specific to mathematics (such as the rules of signs learned in Secondary Cycle One), which ensure that knowledge is constructed in a coherent manner.

[4] *The set of rational numbers includes the set of integers, which includes the set of natural numbers.*

Meaning of operations involving numbers

In order to fully understand operations and their different meanings in various contexts, students must understand the relationships among data and among operations, and choose and perform the correct operations, taking into account the properties[5] *and order of operations. Students must also have a general idea of the result expected.*

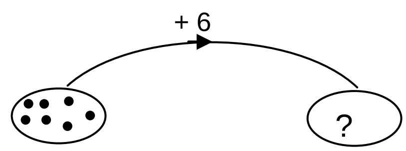
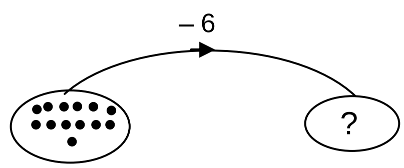
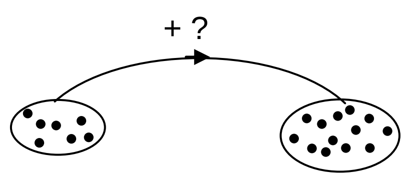
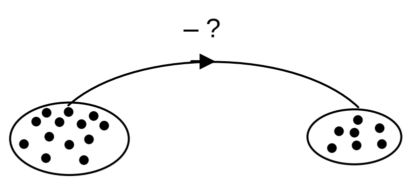
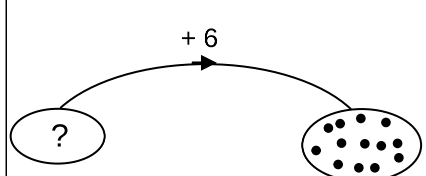
Students will thus be encouraged to use concrete, semi-concrete or symbolic means to mathematize a variety of situations illustrating different meanings. In these situations, students will learn to break problems down into simpler ones and identify the relationships among data that will help them to arrive at a solution. Since operation sense is developed at the same time as number sense, the two should be taught concurrently.

In a spirit of intradisciplinarity, students should have every opportunity to develop and apply their understanding of the different meanings of operations in all branches of mathematics.

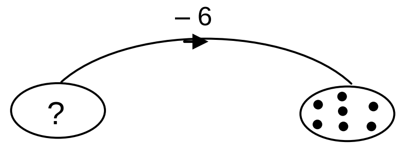
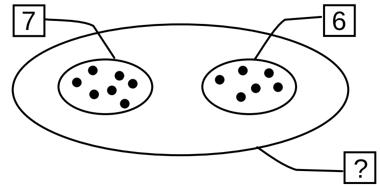
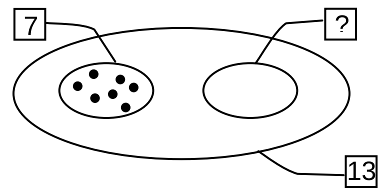
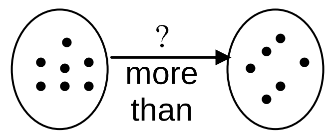
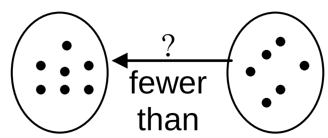
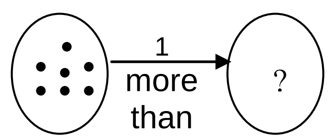
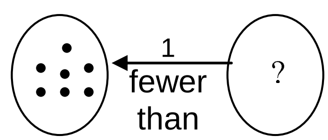
[5] It is advisable to introduce the vocabulary associated with the properties (commutative, associative and distributive) instead of using less precise language to describe them.

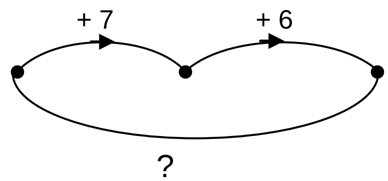
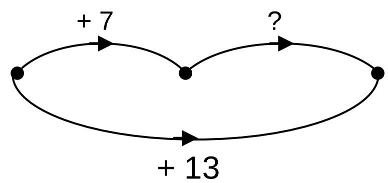
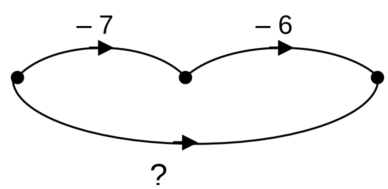
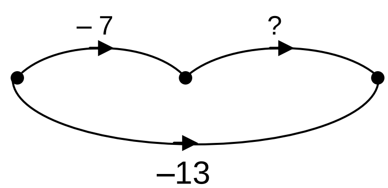
ADDITIVE STRUCTURES

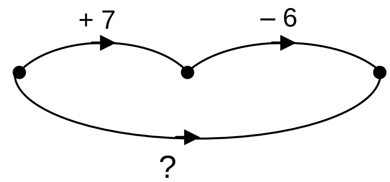
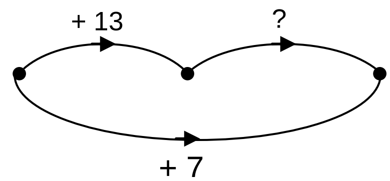
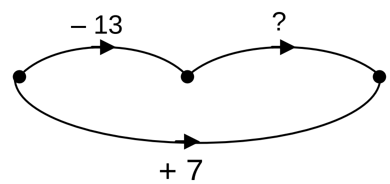
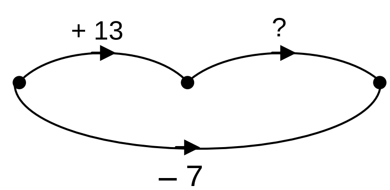
Techniques for performing operations, relationships between operations and the properties of operations only really have meaning when they are used to mathematize situations in order to solve problems. *Additive structures* deal with addition and subtraction, regardless of the type of numbers involved. At each cycle of elementary education, it is essential that a variety of situations be presented: change (adding or taking away), uniting, comparing (more or less than) and combination of changes (positive, negative or mixed). It is not necessary for students to know or memorize the different names for these structures. They must instead develop their own ways of representing them. The table below presents a variety of situations involving varying levels of difficulty.

Structure or meaning	**Situation**[6]	**Model** **(Students create their own representations according to the situation)**	**Equation**
Change (adding) Determine the final state	Gus had 7 objects. Melanie gave him 6 more. How many objects does Gus now have?		7 + 6 = ❐
Change (taking away) Determine the final state	Gus had 13 objects. He gave 6 of them to Melanie. How many objects does Gus have now?		13 – 6 = ❐
Change (adding) Determine the change involved	Gus had 7 objects. Melanie gave him some objects. Gus now has 13 objects. How many objects did Melanie give Gus?		7 + ❐ = 13
Change (taking away) Determine the change involved	Gus had 13 objects. He gave some to Melanie. Gus now has 7 objects. How many objects did Gus give Melanie?		13 – ❐ = 7
Change (adding) Determine the initial state	Gus had some objects. Melanie gave him 6 more. Gus now has 13 objects. How many objects did Gus start with?		❐ + 6 = 13

[6] The following examples consist of only two values each. Teachers should make sure to present situations consisting of several values and involving more than one meaning as well as superfluous or missing data.

Change (taking away) Determine the initial state	Gus had a certain number of objects. He gave 6 to Melanie. He now has 7 objects. How many objects did Gus start with?		❐ – 6 = 7
Uniting Determine the set	Gus has 7 objects. Melanie has 6. How many objects do they have in all?		7 + 6 = ❐
Uniting Determine a subset (complement)	Melanie and Gus have 13 objects all together. Gus has 7. How many objects does Melanie have?		7 + ❐ = 13 13 – 7 = ❐
Comparing (“more than”) Determine the comparison	Gus has 7 objects. Melanie has 6. How many more objects than Melanie does Gus have?		7 = 6 + ❐ 7 – ❐ = 6
Comparing (“fewer than”) Determine the comparison	Gus has 7 objects. Melanie has 6. How many fewer objects than Gus does Melanie have?		7 = 6 + ❐ 7 – ❐ = 6
Comparing (“more than”) Determine a set	Gus has 7 objects. He has 1 more object than Melanie. How many objects does Melanie have?		7 – 1 = ❐ 7 = ❐ + 1
Comparing (“fewer than”) Determine a set	Gus has 7 objects. Melanie has 1 fewer object than Gus. How many objects does Melanie have?		7 – 1 = ❐ 7 = ❐ + 1

Combination of changes (positive) Determine the gain	Yesterday, Gus received 7 objects. Today, he has received 6 more. How many objects has he received over the 2 days?		7 + 6 = ❐
Combination of changes (positive) Determine the change involved	Yesterday, Gus received 7 objects. Today, he has received more but we don't know how many. Given that he has received 13 objects in the past 2 days, has he received more or fewer objects today than he did yesterday? How many objects has he received today?		7 + ❐ = 13
Combination of changes (negative) Determine the loss	Yesterday, Gus gave away 7 objects. Today, he has given away 6. How many objects has he given away in the past 2 days?		–7 – 6 = ❐
Combination of changes (negative) Determine the change involved	Yesterday Gus gave away 7 objects. Today, he has given away some more, but we do not know how many. If in the past 2 days he has given away 13 objects, how many objects has he given away today?		–7 + ❐ = –13

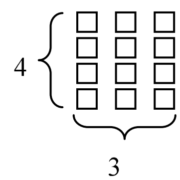
Combination of changes (mixed)[7] Determine the gain or the loss	Gus had a certain number of objects. Yesterday, he received 7 objects. Today, he has given away 6. How many more or fewer objects does he have after 2 days?		7 – 6 = ❒
Combination of changes (mixed) Determine the change involved	Gus had a certain number of objects. Yesterday, he received 13 objects. Given that at the end of 2 days, Gus has 7 more objects than he did initially, how many objects has he received or given away today?		13 – ❒ = 7
Combination of changes (mixed) Determine the change involved	Gus had a certain number of objects. Yesterday, he gave away 13 objects. Given that at the end of 2 days, Gus has 7 more objects than he did initially, how many objects has he received or given away today?		–13 + ❒ = 7
Combination of changes (mixed) Determine the change involved	Gus had a certain number of objects. Yesterday, he received 13 objects. Given that at the end of the 2 days, Gus has 7 fewer objects than he did initially, how many objects has he received or given away today?		13 – ❒ = –7

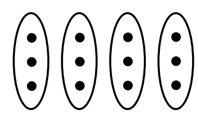
[7] Problems involving a mixed combination of changes require the use of integers. In Elementary Cycle Three, these problems are solved using a diagram or a number line.

MULTIPLICATIVE STRUCTURES

Techniques for performing operations, relationships between operations and the properties of operations only really have meaning when they are used to mathematize situations in order to solve problems. *Multiplicative structures* deal with multiplication and division, regardless of the types of numbers involved. It is much more important to present students with a variety of situations than to emphasize the different names associated with the structures such as: repeated addition, combination or Cartesian product, rectangular arrangement, area and volume, comparison (times as many), repeated subtraction, sharing, number of times x goes into y, and comparison (times fewer than). The table below presents a variety of situations involving varying levels of difficulty.

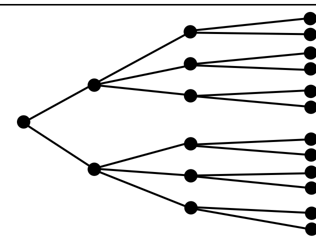
<table>
<tr><th colspan="2">Structure or meaning</th><th>Situation[8]</th><th>Model
(Students create their own representations according to the situation)</th><th>Equation</th></tr>
<tr><td colspan="2">Rectangular arrangement</td><td>In the classroom, there are 3 rows containing 4 desks each.

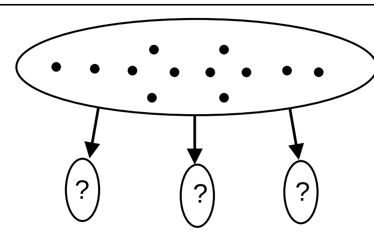
How many desks are in the classroom?</td><td>
</td><td>$3 \times 4 = \square$
or $4 \times 3 = \square$</td></tr>
<tr><td colspan="2">Repeated addition</td><td>Gus receives 3 objects per day.

How many objects will he receive in 4 days?</td><td></td><td>$3 + 3 + 3 + 3 = \square$

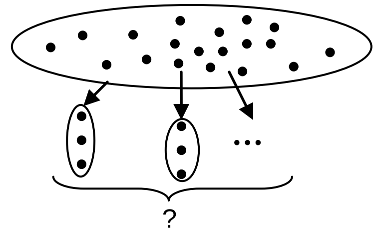
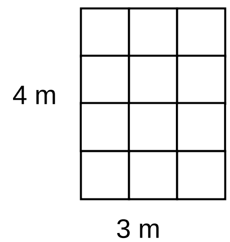
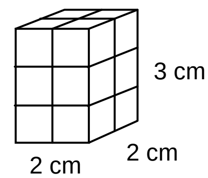
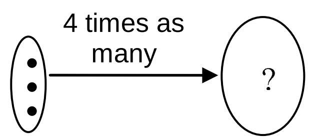
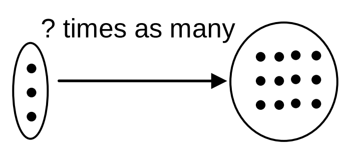
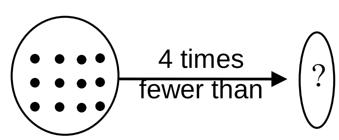
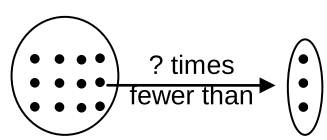
$3 \times 4 = \square$
or $4 \times 3 = \square$</td></tr>
<tr><td rowspan="2">Combination</td><td>Cartesian product</td><td>Gus has 4 shirts and 3 pairs of pants.

How many combinations of pants and shirts can he wear?</td><td><table><tr><th></th><th>S1</th><th>S2</th><th>S3</th><th>S4</th></tr><tr><td>P1</td><td>S1P1</td><td>S2P1</td><td>S3P1</td><td>S4P1</td></tr><tr><td>P2</td><td>S1P2</td><td>S2P2</td><td>S3P2</td><td>S4P2</td></tr><tr><td>P3</td><td>S1P3</td><td>S2P3</td><td>S3P3</td><td>S4P3</td></tr></table></td><td>$4 \times 3 = \square$
or $3 \times 4 = \square$</td></tr>
<tr><td>Tree</td><td>At the cafeteria, there are 2 types of soup, 3 main dishes and 2 desserts on the menu.

How many different meals can you have?</td><td></td><td>$2 \times 3 \times 2 = \square$</td></tr>
<tr><td colspan="2">Sharing</td><td>There are 12 objects in a bag.
They are distributed evenly among 3 friends.

How many objects does each friend receive?</td><td>
</td><td>$12 \div 3 = \square$</td></tr>
</table>

8 The following examples generally consist of only two values each. Teachers should make sure to present situations consisting of several values and involving more than one meaning as well as superfluous or missing data.

Number of times x goes into y	12 objects must be placed in bags. Each bag can hold 3 objects. How many bags are needed?		12 ÷ 3 = ❐
Area	A flower bed is 4 m wide and 3 m long. What is the area of the flower bed?		4 × 3 = ❐ or 3 × 4 = ❐
Volume	A box in the shape of a rectangular prism is 2 cm wide, 2 cm deep and 3 cm high. What is the volume of the box?		2 × 2 × 3 = ❐ 2 × 3 × 2 = ❐ or …
Comparison ("times as many") Determine one of the subsets	Gus has 3 objects. Melanie has 4 times as many objects. How many objects does Melanie have?		3 x 4 = ❐
Comparison ("times as many") Determine the relationship	Gus has 3 objects, and Melanie has 12. How many times more objects than Gus does Melanie have?		3 x ❐ = 12 or 12 ÷ 3 = ❐
Comparison ("times fewer than") Determine one of the subsets	Gus has 12 objects. This is 4 times as many objects as Melanie has. How many objects does Melanie have?		12 ÷ 4 = ❐
Comparison ("times fewer than") Determine the relationship	Gus has 12 objects, and Melanie has 3 objects. How many times fewer objects than Gus does Melanie have?		12 ÷❐ = 3 or 12 ÷ 3 = ❐

Operations involving numbers

As students gradually develop their number and operation sense, they will be called on to develop their own processes, and adopt conventional ones in order to perform various operations. They will learn to recognize equivalencies between these different processes and to develop certain automatic responses. Using these processes and the properties of operations, they will also learn to estimate results and obtain accurate results using mental and written computation.

In order to carry out activities in everyday life, students must develop effective ways to perform mental computation when no aids or tools are available. When students are faced with a situation requiring a calculation, their first decision involves determining if an exact result is required or if an approximate calculation will suffice. The mental skills required to obtain exact or approximate results are developed with practice over the years. Particular attention should be paid to mental computation strategies (often different from written ones), which help students further develop their number and operation sense. Approximations, on the other hand, enable students to verify that their results are relevant, to detect any incorrect results showing on the calculator and to confidently perform calculations. Students should therefore get into the habit of approximating a result before performing a calculation so as to be able to verify the plausibility of the results obtained by other methods.

To transform an equality, students must first be able to describe the different related elements and the relationships among the values. From that equality, they must also be able to derive other equivalent equalities involving the same values, using the properties of equalities, the relationships between the operations and the properties of these operations. (e.g. from the equality 7 + 3 = 10, the following equalities can be generated: 3 + 7 = 10, 10 = 7 + 3, 7 = 10 – 3; 64 + 99 = 64 + 100 – 1, etc.).

The situations presented should involve numerical and non-numerical patterns (e.g. colours, shapes, sounds) to allow students to observe and describe various patterns and series of numbers and operations, such as a sequences of even numbers, multiples of 5 and triangular numbers. These situations will also require students to add terms to a series, state general rules or build models. Thus, students will learn to formulate or deduce definitions, properties and rules.

Mastery of number facts (memorized repertoire) is mainly developed through the use of various materials in different contexts. Students can learn basic number facts by referring to the properties of numbers and operations and patterns such as the use of the number 0 in addition and multiplication, the use of the number 1 in multiplication, counting by multiples (e.g. multiples of 2, of 5), doubles or squares, the commutative law (e.g. 3 + 5 = 5 + 3 = 8), and the distributive law and compensation (e.g. 7×6 is 6 more than 6×6). Students identify these properties or regularities with the help of materials. By observing number tables (in the form of grids), they can more clearly see the general nature of certain properties and discover surprising facts that make them curious as to the reasons behind them. These actions can help with the memorization of number facts and foster students' ability to function automatically and to develop new strategies for remembering number facts that have been forgotten.

In all cycles, calculators may be used to good advantage as a calculation, verification and learning tool (e.g. in situations involving patterns, number decomposition, or the order of operations). They can help students overcome calculation difficulties in different situations so that they can focus on other priorities, such as the development of strategies. Certain meaningful problems can also be covered in class even if the amount of calculations and the size of the numbers involved go beyond the requirements of the program. Calculators also enable certain students to solve more complex problems that focus on the development of strategies rather than on the mastery of written calculations.

INDIVIDUAL PROCESSES

The strategies developed by students through the construction of their own metal or written processes offer several advantages, especially the fact that they are extremely useful in helping to develop number sense. Contrary to conventional processes, which involve the use of digits, individual processes involve the use of numbers.

Written calculation

An important aspect of the mathematics program is that students develop their own processes for written computation. Students have two years to develop them before learning conventional processes: addition and subtraction in Cycle One, multiplication and division in Cycle Two.

When students first record their own processes for written computation, their work is quite unlike conventional computation processes, since it mainly consists of drawings and may involve some symbolism. From the outset, it is important to allow students to explore and discover their own computation processes by helping them make connections with the principals of the base 10 number system. At the beginning of the cycle, they use only drawings to represent their computation processes. Symbolic notation is introduced only toward the end of the cycle, after they have used concrete or semi-concrete manipulatives.

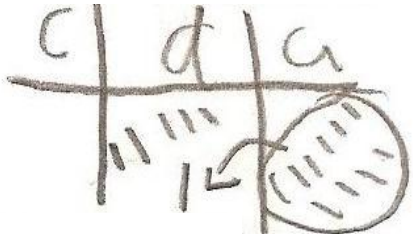
Toward the end of Cycle One, students use both unstructured (tokens, nesting cubes, etc.) and structured (base 10 blocks) manipulatives to carry out operations involving three-digit numbers. These materials enable them to better visualize these operations.

To communicate their computation processes in writing, some students draw each object while others opt for a more symbolic means of representation (e.g. by making less elaborate drawings or using a colour code). These types of methods are completely acceptable coming from students who fully understand the principals of the base 10 number system and who have reached a developmental stage enabling them to select their own means of representation that tend toward symbolism.

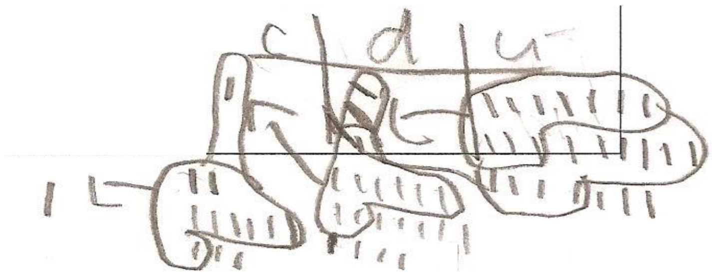
It is important to take into account the needs and progress of each student and not to require that all students represent their computation processes in the same way. These processes should be fully developed by the students themselves and should not merely be a disguised version of a conventional process. After each activity or situation, it is always helpful to discuss the strategies used, since this provides reinforcement for some students and elucidation for others.

Here are some examples[9] of individual processes developed by students in Elementary Cycle One:

1) 26 + 34 = 60

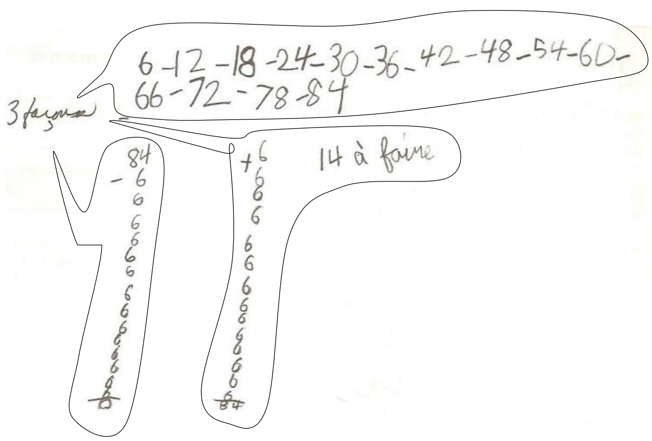


2) 267 + 679 + 347 = 1293

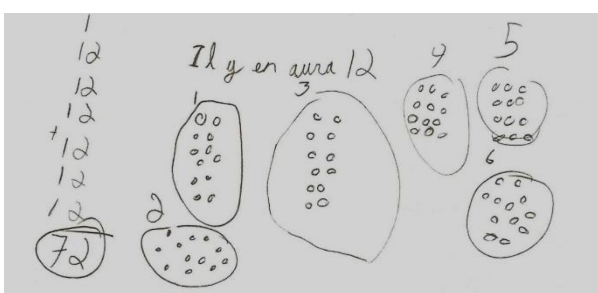


Here are other examples of individual processes developed by students in Elementary Cycle Two:

1) 84 ÷ 6 = 14



2) 72 ÷ 6 = 12



[9] Although students should be encouraged to write operations or equalities across the page, this is not an absolute requirement.

Mental computation

Mental computation is an intellectual activity that is very different from simply memorizing tables. It is much more complex and stimulating. The processes invented by students are often quite ingenious. Students carry out this work in their head, without resorting to paper-and-pencil procedures. For example, to calculate 32 + 59 in writing, students will probably use the following conventional process:

$$\begin{array}{r} \mathbf{1} \\ 32 \\ +\ 59 \\ \hline \mathbf{91} \end{array}$$

When doing the same calculation in their head, it is much more effective to use the following process:

$$\underbrace{32 + \underbrace{60}_{(59\,+\,1)}}_{92\,-\,1\,=\,91}$$

When devising individual processes to perform mental computation, students must interrelate and organize their knowledge about numbers and operations.

For example, to calculate 2×325, I would calculate $\underbrace{2 \times 300}_{600} + \underbrace{2 \times 25}_{50} = 650$.

In this context, number sense is crucial. In particular, students must feel comfortable working with the concepts of place value and number decomposition, finding the order of magnitude and switching from one way of writing numbers to another. Operation sense is also developed as students learn to use the relationships between operations and the properties of operations effectively. The table below presents a few examples of what this involves.

Process	Example	Knowledge
Adding by making the first term a multiple of ten	47 + 14 = 47 + (3 + 11) = (47 + 3) + 11 = 50 + 11 = 61	Place value Decomposition Associative law
Adding by making the first term a multiple of ten Compensation	37 +16 = 37 + 3 + 16 – 3 = 40 + 16 – 3 = 56 – 3 = 53	Associative law
Adding the tens, then adding the ones	49 +28 = 40 + 9 + 20 + 8 = 40 + 20 + 9 + 8 = 60 + 17 = 77	Place value Decomposition Commutative law
Subtracting the tens and then the ones	46 – 12 = 46 – 10 – 2 = (46 – 10) – 2 = 36 – 2 = 34	Place value Decomposition Associative law
Subtracting by making the second term a multiple of ten	54 – 18 = 54 – 20 + 2 = (54 – 20) + 2 = 34 + 2 = 36	Place value Decomposition
Subtracting by first making sure that each term has the same number of units Compensation	51 – 38 = 51 + 7 – 38 – 7 = (58 – 38) – 7 = 20 – 7 = 13	Decomposition

Multiplying by decomposing the multiplicand	$23 \times 4 = (20 + 3) \times 4$ $= (20 \times 4) + (3 \times 4)$ $= 80 + 12$ $= 92$	Decomposition Distributive law
Multiplying by decomposing the multiplier	$23 \times 12 = 23 \times (10 + 2)$ $= (23 \times 10) + (23 \times 2)$ $= 230 + 46$ $= 276$	Decomposition Distributive law
To multiply by 4 or 8, multiply by 2 twice or three times	$13 \times 4 = 13 \times 2 \times 2$ $= 26 \times 2$ $= 52$	Decomposition (into prime factors)
To multiply by 6, multiply by 2, then multiply by 3 or vice versa	$15 \times 6 = 15 \times 2 \times 3$ $= 30 \times 3$ $= 90$	Decomposition (into prime factors)
To multiply by 5, multiply by 10, then divide by 2 or vice versa	$28 \times 5 = 28 \times 10 \div 2$ $= (28 \times 10) \div 2$ $= 280 \div 2$ $= 140$	$28 \times 5 = 28 \div 2 \times 10$ $= (28 \div 2) \times 10$ $= 14 \times 10$ $= 140$
Dividing by making a multiple of the divisor appear in the dividend	$42 \div 3 = (30 + 12) \div 3$ $= (30 \div 3) + (12 \div 3)$ $= 10 + 4$ $= 14$	Decomposition Distributive law
Dividing by making a multiple of the divisor appear in the dividend	$54 \div 3 = (60 - 6) \div 3$ $= (60 \div 3) - (6 \div 3)$ $= 20 - 2$ $= 18$	Decomposition Distributive law
Dividing by breaking down the divisor into several factors	$54 \div 18 = (54 \div 2) \div 9$ $= 27 \div 9$ $= 3$	Decomposition into several factors
To divide by 5, multiply by 2, then divide by 10 or vice versa	$140 \div 5 = 140 \times 2 \div 10$ $= 280 \div 10$ $= 28$	$140 \div 5 = 140 \div 10 \times 2$ $= 14 \times 2$ $= 28$
Compensation	$80 \times 0.3 = (80 \div 10) \times (0.3 \times 10)$ $= 8 \times 3$ $= 24$	
Multiples of the same number Compensation	$3500 \div 500 = (3500 \div 100) \div (500 \div 100)$ $= 35 \div 5$ $= 7$	

GEOMETRY

Before they enter preschool, children explore the shapes of objects in their surroundings and begin to understand basic topological concepts such as inside-outside, above-below; they also acquire the rudiments of spatial sense. In preschool, they begin to organize space and establish relationships between objects by comparing, classifying and grouping them.

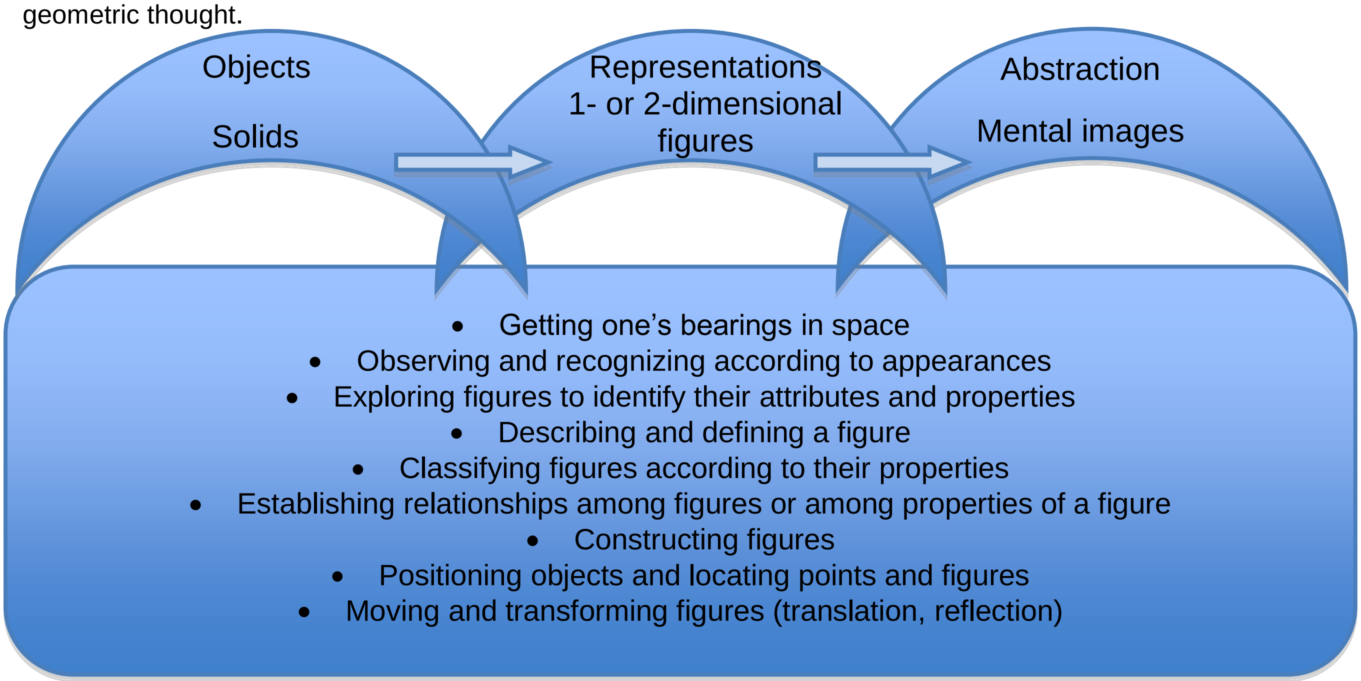
Throughout elementary school, by participating in activities and manipulating objects, students acquire the vocabulary of geometry and learn to get their bearings in space, identify plane figures and solids, describe categories of figures and observe their properties. Geometry in elementary school focuses on two-dimensional (plane) and three-dimensional figures and on key concepts, such as the ability to locate objects in space and observe their geometric and topological properties. Knowledge of vocabulary is not enough; the words must be closely tied to precise concepts such as shape, similarity, dissimilarity, congruency and symmetry. Thus, it is essential for students to use a variety of activities and a wide range of objects in order to develop spatial sense and geometric thought. This will allow students to progress from the concrete to the abstract, first by manipulating and observing objects, then by making various representations, and finally by creating mental images of figures and their properties.

The ability to discern and recognize the properties of a geometric object or a category of objects must be developed before students can learn about the relationships among elements in a figure or among distinct figures. It is also required in order to develop the ability to identify new properties and use known or new properties in problem solving.

The ability to describe various transformations is closely related to other skills, but in itself constitutes a significant element of learning, much like the ability to transform one sequence of arithmetic operations into another. In elementary school, describing a transformation of figures means, being able to recognize and describe reflections and translations. It also means being able to describe a frieze pattern or tessellation in terms of geometric transformations and being able to explain how to use a net in the plane to reconstruct a solid.

Finally, geometry is connected to the other branches of mathematics. By its very nature, geometry is closely related to measurement, provides tools (e.g. circle graph) for conveying information related to statistics and probability, and contributes to the development of number and operation sense through the use of various representations.

The diagram below presents the learning content associated with the development of spatial sense and geometric thought.



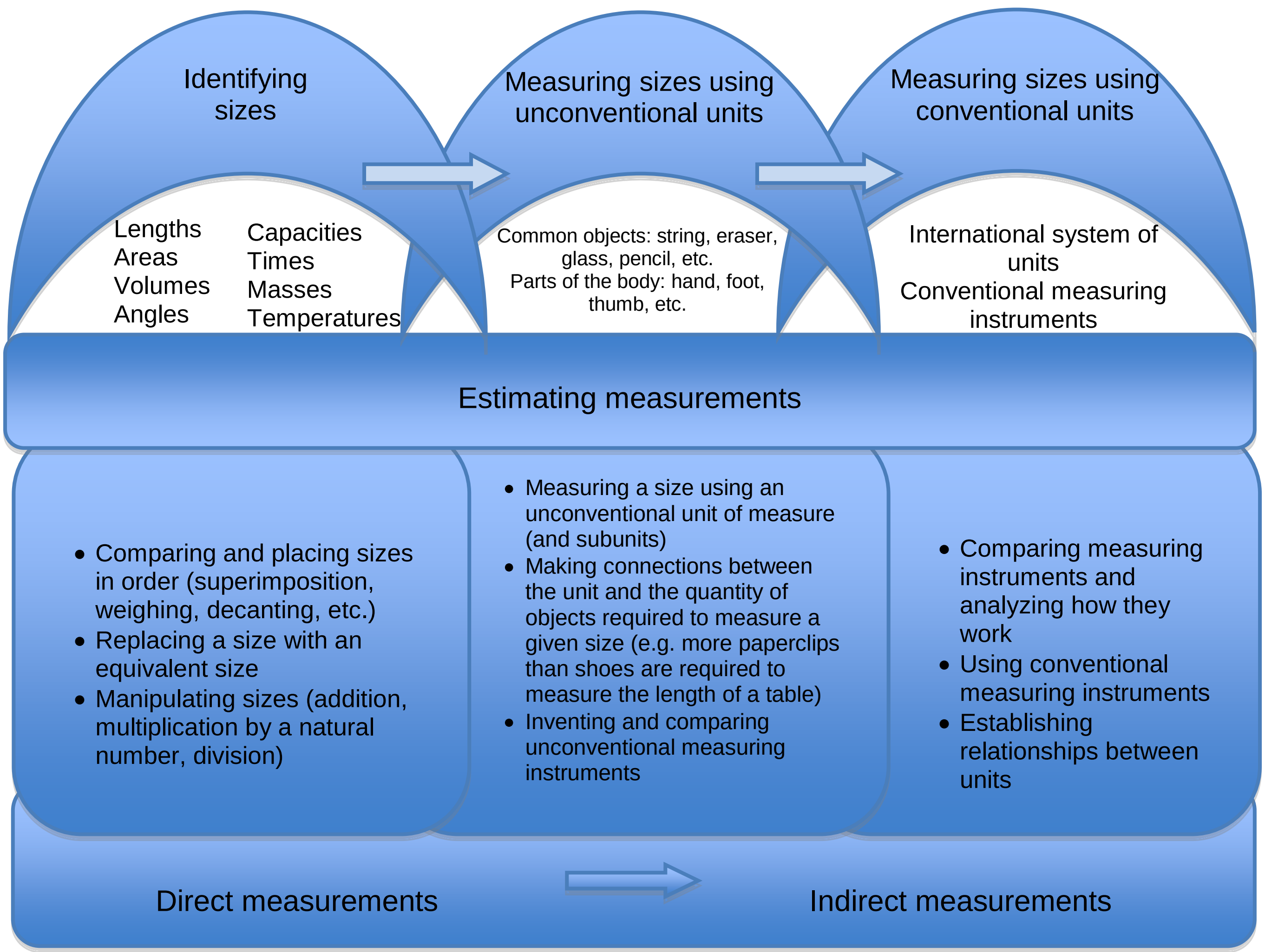
MEASUREMENT

Before they enter preschool, children have acquired the rudiments of measurement in that they have begun to evaluate and compare size. In preschool, they begin to measure things using instruments such as a rope or growth chart.

Establishing a relationship between two geometric figures means recognizing similar shapes or identical measurements (congruence) but also realizing that a figure can fit inside another repeatedly to completely cover it (tessellation, measurement). Measuring therefore involves much more than merely taking a reading on an instrument. Measurement sense is developed by making comparisons and estimates, using a variety of conventional and unconventional units of measure. To develop their sense of measurement (of time, mass, capacity, temperature, angle, length, area and volume), students must participate in activities that allow them to design and build instruments, to use invented and conventional measuring instruments and to manipulate conventional units of measure. They must learn to calculate direct measurements (e.g. calculate a perimeter or area, graduate a ruler) and indirect measurements (e.g. read a scale drawing, make a scale drawing, measure the area of a figure by decomposing it, calculate the thickness of a sheet of paper when the thickness of several pieces is known).

Learning the basic elements of the international system of units help students the material covered in the Science and Technology program. In addition, measurement is related to the other branches of mathematics. It enables students to use their number and operation sense (e.g. base 10, decimal, fractions), spatial sense and knowledge of geometric figures. Measurements are also used in statistics and probability when experiments and surveys are conducted.

The table below presents the learning content associated with the development of measurement sense.



STATISTICS

Throughout elementary school, students participate in conducting surveys[10] to answer questions and draw conclusions. They learn to formulate different types of questions, determine categories or answer choices, plan and carry out data collection, and organize data in tables. To develop statistical thinking, students are thus introduced to descriptive statistics, which allow them to summarize raw data in a clear and reliable (rigorous) way.

By participating in the activities suggested, students will learn to display data using tables, horizontal and vertical bar graphs, pictographs or broken-line graphs, depending on the type of data[11] used. They will also learn to interpret data by observing its distribution (e.g. range, centre, groupings) or by comparing data in a given table or graph. They will ask themselves questions as they compare different questions, samples chosen, the data obtained and their different representations. They will also have the opportunity to interpret circle graphs1[12] and develop an understanding of the arithmetic mean in order to be able to calculate it.

Thus, statistics enables students to draw on their knowledge of arithmetic and measurement and make use of various types of representation, strategies, and reasoning.

PROBABILITY

When attempting to determine the probability of an event, students in elementary school spontaneously rely on intuitive, yet often arbitrary, reasoning. Their predictions may be based on emotions, which may cause them to wish for a predicted outcome or to refute actual results. The classroom activities suggested[13] should help foster probabilistic reasoning. This implies taking into account the uncertainty of outcomes, which may represent a challenge of sorts, since students will tend to determine outcomes by looking for patterns or expecting outcomes to balance out.[14]

In elementary school, students observe and conduct experiments involving chance. They use qualitative reasoning to practise predicting outcomes by becoming familiar with concepts of certainty, possibility and impossibility. They also practise comparing experiments to determine events that are more likely, just as likely and less likely to occur. They list the outcomes of a random experiment using tables or tree diagrams and use quantitative reasoning to compare the actual frequency of outcomes with known theoretical probabilities.

In addition, connections can be made with the other branches of mathematics. For example, statistical tools enable students to record the outcomes of a random experiment in a table and to interpret these results. Connections with arithmetic can be made when enumerating the possible outcomes of an experiment and expressing a probability in numerical form (fraction, percentage, decimal). Geometry and measurement provide opportunities to broaden vocabulary and understand the tools used in probability theory (e.g. dice, spinners) through the application of the properties of geometric figures, for example.

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[10] A survey can involve observations (colours, clothing, shapes, scientific or random experiments), questionnaires, measurements (size, duration), etc.

[11] For example, a broken-line graph is used to illustrate continuous quantitative data (e.g. lengths, temperatures, masses, time)

[12] *Students are not expected to construct circle graphs, but rather to interpret them using the concepts of fraction and percentage.*

[13] The study of probability theory provides a good opportunity to dispel myths regarding chance and encourages the development of critical judgement.

[14] *For example, if the pointer on a two-coloured spinner (red and yellow) stops on yellow three times, students will expect it to land on red when it's their turn.*